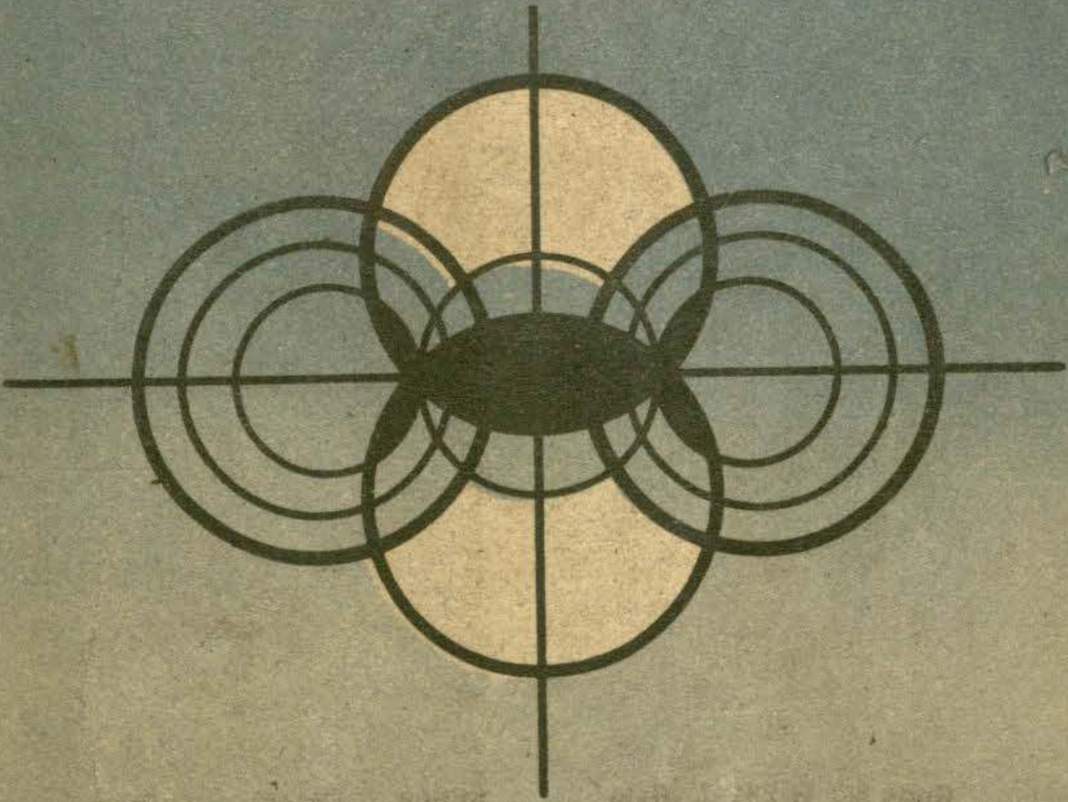
MODERN

CALCULUS AND ANALYTIC GEOMETRY

WITH

MATRICS, DETERMINANTS AND VECTORS

FOR CLASS XII



SARTAJ BOOK DEPOT URDU BAZAR LAHORE

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LAHORE

MODERN

CALCULUS AND ANALYTIC GEOMETRY

WITH

MATRICES, DETERMINANTS AND VECTORS

PUBLISHED BY :-
SARTAJ BOOK DEPOT
6-URDU BAZAR LAHORE
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Chapters II to V, VII, VIII and X of this book are revised and enlarged version of the book entitled
Modern Calculus
with
Analytic Geometry

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PREFACE TO PRESENT EDITION

This book is mainly a revised and enlarged version of the book entitled 'Modern Calculus with Analytic Geometry'. Although it was claimed in the first edition that the approach is modern yet I should admit frankly that it was not so. I may also add that the students and teachers are not yet mentally prepared to accept the most modern approach, based on the Theory of Sets.

In the present edition modern approach is hinted in the Chapter on 'Preliminaries'.

As a consequence of the change in syllabus, the following new chapters are added to the previous edition :

Chapter I	Matrices and Determinants.
Chapter VI	Higher Derivatives and Expansion of Functions.
Chapter IX	Vectors.

The previous edition was full of errors and misprints and a large number of answers of questions were wrong.

I have revised the whole book thoroughly and corrected all the errors. I solved each question individually and its correct answer is given now. During the course of revision some changes and additions were found necessary. These have been incorporated at relevant places.

I think and feel that the teachers and students will certainly find the present edition much better than the previous one.

Any suggestions in respect of the improvement of the book both in quality and content will be most welcome.

LAHORE
22nd July, 1973.

GHULAM KIBRIYA

INTRODUCTION

The main object of this book is to introduce the reader to the fundamental concepts of Analytic Geometry and Calculus.

The Geometry taught in schools is based on methods developed by Euclid, the Greek Mathematician of the third century B.C. It deals with figures like triangles, parallelograms, polygons, circles, conic section and develops their properties by geometrical methods.

The French Mathematician Descartes (1596-1650 A.D.) introduced algebraic methods to study the properties of the above figures. The algebraic methods proved useful in simplifying solutions of many problems which are difficult to solve by Euclid's methods.

The technique developed by Descartes is called Cartesian or Analytic or Co-ordinate Geometry. It depends on a system of co-ordinate axes in plane with which the student is already familiar.

The Calculus is a branch of Mathematics which deals with two main types of problems. The first of these is to find the rate at which a variable quantity is changing. For example, the distance of a moving train from a particular station changes with time and we may ask how fats it is moving at any given instant. Such problems are studied in Differential Calculus.

The second type of problem with which the Calculus is concerned is to find a function when its rate of change is given. Thus, for example, if we are given the velocity of a moving body at every instant of time, we may seek to find the distance it has moved as a function of time. Integral Calculus deals with such problems.

Both branches of Calculus are used in modern science and Engineering as a technique for expressing physical laws in precise mathematical terms and as a tool for studying the consequences of those laws.

We shall see that Analytic Geometry helps in understanding the basic concepts of Calculus, and Calculus helps in understanding the concepts of Analytic Geometry. Thus Calculus and Analytic Geometry can be united and studied as a whole.

PREFACE TO FIRST EDITION

This book has been prepared for the second year students of the Intermediate stage in the Colleges of Pakistan under the scheme to *modernise* the educational system of the country sponsored by the Government of Pakistan.

Under the old system Analytical Geometry and Calculus were developed independent of each other. Consequently there was lot of repetition, e.g., tangents, normals, intersection of curves were first tught in Analytical Geometry and then repeated in Calculus. This *out-dated* approach has been discarded throughout the world.

In the present textbook we have made an endeavour to approach the subjects of Analytical Geometry and Calculus in the *modern* way. We have developed the two subjects side by side so that each may assist the other. This integration of Analytical Geometry and Calculus has simplified their study a great deal and has enhanced their utility.

In the first two chapters, we have developed those topics of Analytical Geometry which do not fall under the jurisdiction of Calculus. In the third chapter fundamentals of Calculus are discussed. Ideas of limit and continuity are introduced in a simple way. Concepts of rate of change and derivative are also included in this chapter. The fourth chapter is on differentiation of algebraic functions and its applications to tangents and normals of conic sections and other curves. The fifth chapter discusses the applications of Differential Calculus to maxima and minima, motion of a particle and allied problems. The sixth chapter deals with concepts of definite and indefinite integrals and their applications. Differentiation and integration of trigonometric and exponential functions are discussed in the seventh chapter in an integrated form. Eighth Chapter gives different methods of Integration.

Throughout the book examples are selected from Geometry and Mechanics. Thus, for example, it is explained why search lights or antennas take the shape of paraboloids. Again it is explained how a weak sound at one place might be heard more distinctly at another place even though it might be inaudible to some one much closer to the former place. Such examples are bound to develop the interest of the students in the subject who will no longer regard mathematics as a set of uninteresting rules for computational work.

Suggestions for the improvement of the book will be welcome.

We thank the members of the Review Committee who went through the manuscript minutely and made many valuable suggestions for the improvement of the book. Considerable portion of the book was rewritten according to the suggestions of the Committee.

We also thank the Goveanment of Pakistan, and the authorities of all the Universities and of the Boards of Intermediate and Secondary Education, Pakistan, who have given all possible help in the preparation of this book.

S. M. YUSUF

PESHAWAR :
1st February, 1968.

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x

NOTATION

$<$	is less than
$>$	is greater than
$\leqslant$	is less than or equal to
$\geqslant$	is greater than or equal to
$\nless$	is not less than
$\ngtr$	is not greater than
$=$	is equal to
$\neq$	is not equal to
$n!$	n factorial $=n(n-1)(n-2)\cdots\cdots 3.\ 2.\ 1$
$\vert x \vert$	absolute value of x
$\sim$	is similar to
$\parallel$	is parallel to
$\in$	belongs to
$\subset$	is contained in
iff	if and only if
$\Rightarrow$	Implies
w.r.t	with respect to
§	article (or section)

CHAPTER 0

PRELIMINARIES

0.1. *Field Properties of Real Numbers*—What is a real number? This is a natural question, since the study of Determinants, Matrices, Vectors, Calculus and Analytic Geometry presumes a knowledge of real number. We are not in a position to answer this question directly at this stage since we would have to study certain other concepts such as natural numbers, integers and rational numbers. All these number systems have, to some extent, been dealt with previously without of course formally defining them. The fact is that we need not answer the above question directly, *i.e.*, we can avoid defining a real number formally. As a matter of fact we have described the set R of real numbers in 'Modern Algebra and Trigonometry' and this description is enough to distinguish the set of real numbers from other sets.

In this book the set of real numbers will be understood to be a *field** having a *linear order relation*. The term "linear order relation" is a highly technical term, which will be defined in §0.4, so that its concept is clearly understood.

0.2. *Coordinate Systems*—Coordinate systems are basically devices to locate position of an object. Some everyday examples of coordinate systems are the following :

* A house is located by its street and number.

* A temperature is indicated by a number above or below a fixed zero point on a thermometer.

* A hip is located by its longitude and latitude on the earth's surface.

---

*For the study of fields, consult 'Modern Algebra and Trigonometry.

Without going into the technical details, let us agree that 1-dimensional space (briefly, 1-space) needs just one coordinate to specify the position of a point in it, whereas a point in 2-space will be specified by two labels, 3-space by three labels and so on.

In this book our work is confined to the study of 1-space (the "*line*"), 2-space (the "*plane*") and 3-space, and in each case *position* of a point in the relevant space will be indicated by a single real number, an ordered pair of real numbers and an ordered triple of real numbers respectively.

0.3. *Mathematical Induction and Natural Numbers.*

Definition—A subset S of the set of real numbers is said to be an *inductive set*

$$\Leftrightarrow (i)\ 1 \in S \text{ and}$$
$$(ii)\ x \in S \Rightarrow x+1 \in S$$

It looks obvious that the set of all natural numbers is an inductive set. There are many other inductive sets. For example the set of integers, rational numbers and real numbers are all inductive sets. From the above definition it is clear that

$$1,\ 1+1,\ (1+1)+1,\ [(1+1)+1]+1,\ \ldots\ldots\ldots\ldots$$

are members of every inductive set. Thus we are led to define the natural numbers as follows :

Definition of Natural Number. A real number x is a natural number $\Leftrightarrow$ x is a member of every inductive set.

In other words the set of natural numbers consists of those real numbers which are members of every inductive set.

From this definition it is also clear that the set of natural numbers is the smallest inductive subset of the set of real numbers. That is, the set of natural numbers is itself an inductive set and is included in all other inductive sets. This consequence of the definition of natural number is one of the most powerful tools in mathematics. This is called the *Principle of Mathematical (or complete) Induction.*

Formally it is stated as :

If S is an inductive set, and x is a natural number, then $x \in S$.

0.4. *Partially Ordered Sets.*—If a binary relation R is defined on a set S with the properties that :

$$\left.\begin{array}{l} (i)\ \text{(reflexive property)}\quad (a, a) \in R \\ (ii)\ \text{(antisymmetric property)} \\ \qquad\qquad (a, b), (b, a) \in R \Rightarrow a = b, \\ (iii)\ \text{(transitive property)} \\ \qquad\qquad (a, b), (b, c) \in R \Rightarrow (a, c) \in R \end{array}\right\} \forall\ a, b, c \in S,$$

then S is called a ***partially ordered set*** and R is called a *partial order* on S

If, in addition to the above properties (*i*)—(*iii*), R satisfies the following property :

(*iv*) $\forall\ a, b \in S$, either $(a, b) \in R$ or $(b, a) \in R$ or $a = b$, then S is called a *fully ordered* (or *linearly ordered* or *totally ordered*) set and R in turn is known as the *full order* (or *linear order* or *total order*).

Sometimes (for the sake of convenience) we drop the word "full" and call S an *ordered set*) and R is known as the *order*.

0.5. *Linear Ordering of Real Numbers.* To represent the set of points on a line (1—space) all that is needed is a point of reference, say O (called the origin), which may be selected arbitrarily. Then any point on the line can be located simply by indicating the distance of that point from O and the side of O on which the point lies. We agree to label points on the right of O by the real numbers which indicate the distance from O, whereas points on the left of

O are labelled by the additive inverses of the distances from O as shown in figure 1.

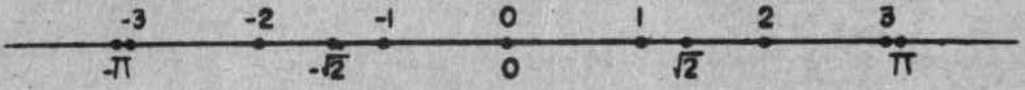


figure 1

In this way we are assigning a unique real number to correspond to each point in 1-space. This correspondence between the points of the line (1-space) and the set of real numbers is a 1—1 correspondence. Accordingly it is permissible and true to describe the position (of an element) in 1-space either as a *point* or as a *real number*. This 1—1 correspondence accounts for the common name *real number line*, or simply *real line* for 1-space. The real number which corresponds to a particular point on this line is said to be the coordinate of that point. Informally we may regard coordinates as addresses of the points, since they (the coordinates) indicate the positions of the points, on the real line.

The points in 1—space give a geometric view of the so called " linear ordering " of the real number, in the sense that we may establish an *order relation* between any two real numbers. In symbols we write $<$ to mean " precedes " (for a geometric view) and " is less than " (for an algebraic aspect). Then the opposite orientation, $>$ means "follows" (geometrically) and " is greater than" (algebrically). Accordingly we say :

$a<b$ means "a precedes b" or "a is less than b"

$2<5$ means "2 precedes 5" or "2 is less than 5".

$-3<0$ means "-3 precedes 0 or "-3 is less than 0"

and

$b>a$ means "b follows a" or "b is greater than a"

$5>2$ means "5 follows 2" or "5 is greater than 2"

$0>-3$ means "0 follows-3" or "0 is greater than-3"

It is to be noted that the symbol $>$ is only used for the sake of convenience, since the statement $b>a$ is just another way of writing $a<b$. It is also customary to read $a \leqslant b$ as "a is less than or equal to b" and $a < b$ as a is strictly less than b."

Although the idea of ordering the real numbers arises naturally in a geometric way, nevertheless we avoid to give a geometric picture and instead follow the logical development, since the effect of the linear order relation on the field of real numbers is completely described by the following four simple axioms $A_1 - A_4$..

(A_1) *Law of Trichotomy.*

For every real number x, one and only one of the following relations is true.

either $x > 0$ or $x = 0$ or $x < 0$.

(A_2) *Closure of Positive Real Numbers Under Addition.*

For every pair of real numbers x and y,

$x>0, y>0 \Rightarrow x+y>0$.

(A_3) *Closure of Positive Real Numbers Under Multiplication.*

For every pair of real numbers x and y,

$x > 0, y > 0 \Rightarrow xy > 0$

(A_4) *Ordering by Positivity.*

For every pair of real numbers x and y,

(1) $x < y \Leftrightarrow y - x > 0$

(2) $y > x \Leftrightarrow x < y$

0.6. *Definition of Negative Real Number.*

A real number x is called a ***negative*** real number if and only if $-x > 0$ (or equivalently $x < 0$). In other words we may say that a ***negative*** real number is a number whose additive inverse is positive. We now state propositions $P_1 - P_{11}$ concerning the real numbers

(P_1) All natural numbers are positive, that is for every natural number n,

$$n > 0$$

(P_2) For every real number x,

$$x > 0 \Rightarrow x^{-1} > 0$$

(P_3) For every real number $x(\neq 0)$,

$$x^2 > 0$$

(P_4) For every pair of real numbers x, y,

$$x > 0 \text{ and } y < 0 \Rightarrow xy < 0$$

(P_5) For every pair of real numbers x, y,

$$x < 0 \text{ and } y < 0 \Rightarrow xy > 0$$

(P_6) For every pair of real numbers x, y $(\neq 0)$

$$0 < x < y \Leftrightarrow 0 < y^{-1} < x^{-1}$$

or equivalently

$$0 < x < y \Leftrightarrow x^{-1} > y^{-1} > 0$$

(P_7) For every triple x, y, z, of real numbers,

$$x < y \text{ and } y < z \Rightarrow x < z$$

or dually

$$z > y \text{ and } y > x \Rightarrow z > x$$

(P_8) For every triple x, y, z of real numbers

$$(i)\ x < y \text{ and } z > 0 \Rightarrow xz < yz.$$
$$(ii)\ x < y \text{ and } z < 0 \Rightarrow xz > yz.$$

(P_9) For every triple x, y, z of real numbers,

$$x < y \Rightarrow x+z < y+z$$

(P_{10}) For ever set of four real numbers x, y, z, t,

$$x < y \text{ and } z < t \Rightarrow x+z < y+t$$

(P_{11}) For every set of four real numbers x, y, z, t

$$0 < x < y \text{ and } 0 < z < t \Rightarrow 0 < xz < yt.$$

The above propositions are silent about subtraction and division. The reason is that we can convert problems about subtraction and division into problems involving addition and multiplication simply by writing

$$x-y=x+(-y) \text{ and}$$
$$\frac{x}{y}=xy^{-1}\ (y\neq 0).$$

Note: It should be noted that 0 is neither positive nor negative.

CHAPTER I

MATRICES AND DETERMINANTS

1.1. *Introduction.*—The notation of ordinary algebra is a convenient, compact and well defined code for expressing the logical relations of numbers. Initially the notation of matrices was merely a more convenient and shorthand system of tackling various problems of algebra. The rules of operation are so few and simple, yet so general and profitable, that matrices are now used in almost all the branches of Mathematics. Matrices and determinants are especially useful in the solution of systems of linear equations. It is, therefore, logical to start this book with a brief survey* of matrices and determinants.

1.2. *Matrix.*—A rectangular array (or table) of numbers** or functions enclosed by a pair of brackets and subject to certain rules is called a *matrix*.

$$\begin{bmatrix} 3 & 7 & 1 \\ 4 & 2 & 5 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 1/2 & \sqrt{3} & 1 \\ 0 & -5 & 2 \\ 3 & 4 & -1/3 \end{bmatrix}$$

are examples of matrices.

In general an array, such as

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots\ldots a_{2n} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots \\ \ldots & \ldots & \ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} \ldots\ldots\ldots a_{mn} \end{bmatrix}$$

---

* The student may have just a glimpse of the subject, since only a small portion of theory is included in the Intermediate syllabus.

** In this chapter we shall only speak about real numbers.

is a matrix of *order* $m\times n$ (read as "m by n"). The numbers or functions a_{ij} ($i=1, 2, \ldots\ldots, m$ and $j=1, 2, \ldots\ldots, n$) are called the *elements* or *entries* of the matrix. The horizontal arrays namely

$$a_{11}\ \ a_{12}\ \ a_{13}\cdots\cdots a_{1n}\ ;\ a_{21}\ a_{22}\ \ a_{23}\cdots\cdots a_{2n}\ ;\ \cdots\cdots\cdots\cdots\ ;\ a_{m1}\ \ a_{m2}\ \ a_{m3}\cdots\cdots\cdots a_{mn}$$

are known as the *rows* of the matrix M, whereas the vertical arrays, such as

$$\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ \vdots \\ a_{m1} \end{array} \qquad \begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ \vdots \\ a_{m2} \end{array}\ ;\ \cdots\cdots\cdots\ ;\ \begin{array}{c} a_{1n} \\ a_{2n} \\ \vdots \\ \vdots \\ a_{mn} \end{array}$$

are said to be the *columns* of the matrix M. If $m=n$, that is, if the number of rows is equal to the number of columns, the matrix is called a *square* matrix.

For example

$$\begin{pmatrix} 2 & 1 \\ \frac{1}{2} & -\frac{1}{3} \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{2} & 0 & 3 \\ -1 & 1 & 7 \\ 2 & 4 & -5 \end{bmatrix}$$

are square matrices of order 2 and 3 respectively. It is to be noted that if A is a square matrix having n rows and n columns then it is customary to call it a matrix of order n instead of a matrix of order $n\times n$.

1.3. *Notation.*—Matrices are usually indicated by capital letters A, B, C, ……… L, M, N, ………, X, Y, Z and entries by small letters $a, b, c, \ldots\ldots, l, m, n, \ldots\ldots, x, y, z$ of the Roman (English) alphabet. In the developement of the theory of matrices the double suffix notation such as $a_{11}, a_{12}, a_{13}, \ldots\ldots a_{21}, a_{22}, \ldots\ldots$ is found very convenient and useful. It is also sometimes convenient to denote a matrix by writing a *typical* entry within brackets and by specifying the order of the matrix. For example the matrix M of § 1.2 may be written as

$$M=[a_{ij}] \qquad (i=1, 2, \ldots\ldots, m;\, j=1, 2, \ldots\ldots, n).$$

Examples. Write the following matrices in tabular form.

(*i*) $[a_{ij}]$, $i=1, 2, 3$; $j=1, 2$.

(*ii*) $[a_{ik}]$, order 3×3

(*i*) $[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ (*ii*) $[a_{ik}]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Definition. Two matrices *A* and *B* of the same order are said to be *equal* if every entry of *A* is equal to the corresponding entry of *B*.

if $A=[a_{ij}]$ and $B=[b_{ij}]$, then
$A=B \Longleftrightarrow a_{ij}=b_{ij} \ \forall\ i, j.$

Exmples :

(*i*) If $A=\begin{bmatrix} 2 & 3 & 4 & 7 \\ 1 & 0 & 0 & 9 \end{bmatrix}$ and

$$B=\begin{bmatrix} 1+1 & 4-1 & 4 & 9-2 \\ 8-7 & 2-2 & 3-3 & 5+4 \end{bmatrix}$$

then $A=B$.

(*ii*) If $A=[a_{ij}]$ and $B=[b_{ij}]$ are two matrices of order 3×3, then $A=B \iff a_{11}=b_{11}\ \ a_{12}=b_{12}\ \ a_{13}=b_{13}$ $a_{21}=b_{21}, a_{22}=b_{22}, a_{23}=b_{23}, a_{31}=b_{31}, a_{32}=b_{32}$ and $a_{33}=b_{33}$.

1.4. *Addition of Matrices*. If $A=[a_{ij}]$ and $B=[b_{ij}]$ are two matrices of order $m\times n$, their *sum* $A+B$ is defined to be the matrix $C=[c_{ij}]$, where each element of C is the sum of the corresponding entries of A and B.

In symbols

$$C=A+B=[a_{ij}+b_{ij}]$$

The following things are evident from this definition.

(*i*) The matrices A and B can only be added if order of A = order of B.

(*ii*) order of A = order of B = order of A+B.

This definition can be extended to the addition of any number of matrices of the same order. If two matrices have the same order, they are said to be *conformable for addition.*

Example. Which of the following matrices are conformable for addition.

$$A=\begin{bmatrix}1 & 2 & 3\\ -2 & -1 & 4\\ 0 & 5 & 0\end{bmatrix},\quad B=\begin{bmatrix}0 & 0 & 3\\ 1 & 0 & 7\end{bmatrix},\quad C=\begin{bmatrix}4 & 5\\ 6 & -7\end{bmatrix}$$

$$D=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad E=\begin{bmatrix}\sqrt{2} & \sqrt{3} & 6\\ -9 & 2 & 1\end{bmatrix}$$

$$F=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix},\quad G=\begin{bmatrix}2 & 3\\ 1 & 4\\ 5 & 7\end{bmatrix}$$

The matrices A and F, B and E, C and D are conformable for addition ; whereas

(*i*) G is not conformable with any of the matrices A, B, C, D, E or F.

(*ii*) A and F are not conformable with any of the matrices B, C, D, E or G.

(*iii*) B and E are not comformable with any of the matrices A, C, D, F or G.

(*iv*) C and D are not comformable with any of the matrices A, B, E, F or G.

1.4.1. *Definition of a Null Matrix.* A matrix O of order $m \times n$ having all zero entries is called a null matrix (zero matrix) of order $m \times n$.

$$O_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is a null matrix of order } 2\times 2,$$

$$O_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is a null matrix of order } 2\times 3,$$

$$O_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is a null matrix of order } 3\times 2 \text{ and so on.}$$

Example

Let $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 2 & \frac{1}{2} & 5 \\ 1 & 0 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & -3 & 6 \\ 4 & 2 & 1 \end{bmatrix}$. Find out which of the matrices A, B, C, D are conformable for addition with O.

Order of O = order of C = order of D = 2×3.

Each of the matrices C and D is conformable for addition with O.

$$\text{Clearly } O+C=\begin{bmatrix} 0+2 & 0+\frac{1}{2} & 0+5 \\ 0+1 & 0+0 & 0+3 \end{bmatrix}=\begin{bmatrix} 2 & \frac{1}{2} & 5 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= C=C+O$$

$$\text{And } C+O=\begin{bmatrix} 2+0 & \frac{1}{2}+0 & 5+0 \\ 1+0 & 0+0 & 3+0 \end{bmatrix}=\begin{bmatrix} 2 & \frac{1}{2} & 5 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= C_1 \therefore O+C=C+O$$

Similarly $O+D=D=D+O$

By the above argument we can prove that

Rule 1. If A is a matrix of order $m \times n$ and O is a null matrix of order $m \times n$, then

$$A+O=O+A=A$$

1.4.2. *Definition.* If $A=[a_{ij}]$ is any matrix, then $-A$ (minus A) is defined to be the matrix having the same entries as A but with opposite signs. Thus

$$-A=[-a_{ij}]$$

Example :

$$(i) \text{ If } A=\begin{bmatrix} 2 & 3 & -1 \\ -3 & 0 & 7 \end{bmatrix} \text{ then } -A=\begin{bmatrix} -2 & -3 & 1 \\ -2 & 0 & -7 \end{bmatrix}$$

$$(ii) \text{ If } B=\begin{bmatrix} 0 & 0 & -7 \\ 2 & -1 & 4 \\ 6 & 5 & \frac{1}{2} \end{bmatrix}, \text{ then } -B=\begin{bmatrix} 0 & 0 & 7 \\ -2 & 1 & -4 \\ -6 & -5 & -\frac{1}{2} \end{bmatrix}$$

(*iii*) If

$$C=\begin{bmatrix} c_{11} & c_{12} & -c_{13} \\ c_{21} & -c_{22} & c_{23} \\ -c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{42} \end{bmatrix}, \text{ then} -C=\begin{bmatrix} -c_{11} & -c_{12} & c_{13} \\ -c_{21} & c_{23} & -c_{23} \\ c_{31} & -c_{22} & -c_{33} \\ -c_{41} & -c_{42} & --c_{43} \end{bmatrix}$$

From the above examples it is clear that :

Rule 2. If $A=[a_{ij}]$ is any matrix of order $m \times n$, then $A-A=[a_{ij}-a_{ij}]=O$, where O is a null matrix of order $m \times n$.

Example. *Add* the following matrices.

(i) $\mathbf{X}=\begin{bmatrix}3 & 7 & -9\\ 1 & 2 & 4\end{bmatrix}, \mathbf{Y}=\begin{bmatrix}2 & 5 & 0\\ 10 & -7 & 6\end{bmatrix}$

(ii) $\mathbf{L}=\begin{bmatrix}l_{11} & l_{12} & l_{13}\\ l_{21} & l_{22} & l_{23}\\ l_{31} & l_{32} & l_{33}\end{bmatrix}$

$$\mathbf{M}=\begin{bmatrix}m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{bmatrix}$$

(i) $\mathbf{X}+\mathbf{Y}=\begin{bmatrix}3+2 & 7+5 & -9+0\\ 1+10 & 2-7 & 4+6\end{bmatrix}$

$$=\begin{bmatrix}5 & 12 & -9\\ 11 & -5 & 10\end{bmatrix}=\mathbf{Y}+\mathbf{X}$$

(iii) $\mathbf{L}+\mathbf{M}=\begin{bmatrix}l_{11}+m_{11} & l_{12}+m_{12} & l_{13}+m_{13}\\ l_{21}+m_{21} & l_{22}+m_{22} & l_{23}+m_{23}\\ l_{31}+m_{31} & l_{32}+m_{32} & l_{33}+m_{33}\end{bmatrix}$

$$\mathbf{M}+\mathbf{L}=\begin{bmatrix}m_{11}+l_{11} & m_{12}+l_{12} & m_{13}+l_{13}\\ m_{21}+l_{21} & m_{22}+l_{22} & m_{23}+l_{23}\\ m_{31}+l_{31} & m_{32}+l_{32} & m_{33}+l_{33}\end{bmatrix}$$

Hence

Rule 3.—The sum of two matrices is commutative if and only if their corresponding entries commute among themselves.

It is however pointed out that the entries of a matrix are usually the elements of some field F in which addition is commutative, hence in such cases for any two matrices A and B of the same order we have

$$A+B=B+A.$$

The reader should note carefully that from now onwards all the matrices under consideration are supposed to have entries belonging to some field F.

1.5. *Definition* (*Scalar Multiplication*).—Suppose that A is a matrix of order $m\times n$ and c is any real number. The matrix c A obtained by multiplying each of the entries of the matrix A by the number c is called the *scalar multiple* of A. In symbols

$$c\,A=c[a_{ij}]=[ca_{ij}].$$

The process itself is known as the *Scalar Multiplication.* It is obvious from the definition that

order of A = order of c A.

Examples

(*i*) If $A=\begin{bmatrix} 5 & -6 & 1 & 4 \\ \frac{1}{2} & 0 & -\sqrt{3} & 2 \end{bmatrix}$, then

$$7A=\begin{bmatrix} 7(5) & 7(-6) & 7(1) & 7(4) \\ 7(\frac{1}{2}) & 7(0) & 7(-\sqrt{3}) & 7(2) \end{bmatrix},$$

$$=\begin{bmatrix} 35 & -42 & 7 & 28 \\ \frac{7}{2} & 0 & -7\sqrt{3} & 14 \end{bmatrix},$$

$$\sqrt{2}\,A=\begin{bmatrix}5\sqrt{2} & -6\sqrt{2} & \sqrt{2} & 4\sqrt{2}\\ \frac{1}{\sqrt{2}} & 0 & -\sqrt{6} & 2\sqrt{2}\end{bmatrix}$$

and $-\frac{1}{5}A=\begin{bmatrix}-1 & \frac{6}{5} & -\frac{1}{5} & -\frac{4}{5}\\ -\frac{1}{10} & 0 & \frac{\sqrt{3}}{5} & -\frac{2}{5}\end{bmatrix}$

(*ii*) If $X=\begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{bmatrix}$ and r is any real number, then

$$r\,X=\begin{bmatrix}rx_{11} & rx_{12} & rx_{13}\\ rx_{21} & rx_{22} & rx_{23}\\ rx_{31} & rx_{32} & rx_{33}\end{bmatrix}$$

Two matrices A and B are called *conformable for multiplication 'AB'* if the number of columns in A is equal to the number of rows in B.

Examples

(*i*) If $A=\begin{bmatrix}9 & 3 & -1\\ 2 & 4 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ 2 & 0\\ 0 & -3\end{bmatrix}$; then

number of columus in $A=3$

number of rows in $B=3$.

Thus the matrices A and B are conformable for multiplication AB.

Again

number of columns in $B=2$
number of rows in $A=2$

Thus the matrices A and B are also conformable for multiplication BA.

$$(ii)\ \text{If } A=\begin{bmatrix}1 & \frac{1}{2} & 4\\ 2 & 3 & 7\end{bmatrix} \text{ and } B=\begin{bmatrix}3 & -3 & \sqrt{3}\\ 4 & 5 & 1\\ 2 & 0 & 0\\ 1 & -1 & 6\end{bmatrix}$$

then

number of columns in $A=3$,
number of rows in $B=4$.

Thus the matrices A and B are *not* conformable for multiplication AB

Again

number of columns in $B=3$,
number rows in $A=2$

Thus the matrices A and B are *not* comformable for multiplication BA.

$$(iii)\ \text{Let } X=\begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\\ x_{41} & x_{42} & x_{43}\end{bmatrix} \text{ and } Y=\begin{bmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\\ y_{31} & y_{32}\end{bmatrix},$$

Now number of columns in X $=$number of rows in $Y=3$. Hence X and Y are conformable for multiplication XY.

On the other hand,

number of columns in $Y=2$,
number of rows in $X=4$.

Hence X and Y are not conformable for multiplication YX. It is, therefore, evident from these examples that given any two matrices A, B it may be possible to form

- **(*a*) both AB and BA,**
- **(*b*) AB or BA,**
- **(*c*) neither AB nor BA.**

The product AB (if possible) is known as the *pre-multiplication* of B by A, whereas the product BA (if possible) is called the *post-multiplication* of B by A.

If A and B are square matrices of the same order then it is always possible to form both the products AB and BA.

1.6. *Multiplication of Matrices*.—If A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$ then their product AB is defined as the matrix whose element in the *i*th row and *j*th column is obtained by summing the products of the corresponding entries in the *i th* row of A and in the *j*th column of B.

In symbols, $k = 1, 2, \ldots\ldots n$

if $A = [a_{ik}]$ and $B = [b_{kj}]$, then $i = 1, 2, 3 \ldots . m$

$AB = [a_{i1}\, b_{ij} + a_{i2}\, b_{2j} + \ldots\ldots\ldots + a_{in}\, b_{nj}]$ $j = 1, 2, \ldots\ldots p$

$$= [\sum_{k=1}^{m} a_{ik}\, b_{kj}]$$

The matrix AB is of order $m \times p$.

In particular, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}. \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}, \text{ then}$$

AB =

$$\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34} \end{bmatrix}$$

which is a matrix of order 2 × 4

Explanation. To form the product AB we proceed as follows :

(*i*) Multiply the corresponding entries in the first row of A and in the first column of B and add them. This is the entry in the first row and first column of the matrix AB.

(*ii*) Multiply the corresponding entries in the first row of A and in the second column of B and add them. This is the entry in the first row and second column of the matrix AB.

(*iii*) Multiply the corresponding entries in the first row of A and in the third colunm of B and add them. This is the entry in the first row and third column of the matrix AB.

(*iv*) Multiply the corresponding entries in the first row of A and in the fourth column of B and add them. This is the entry in the first row and fourth column of the matrix AB.

Since all the columns of the matrix B have been exhausted, it is an indication that we have completed the first row of the matrix AB.

To complete the second row of the matrix AB, we multiply the entries in the second row of A to the corresponding entries in the first, second, third and fourth columns respectively of B and add them. This completes the process.

Example 1. Multiply the following matrices :

$$X=[x_1 \quad x_2 \quad x_3],\ Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

Number of columns in X = 3
= number of rows in Y.

Thus we can form the product XY as follows :

$XY=[x_1y_1+x_2y_2+x_3y_3]$, which is a matrix of order 1×1 *i.e.*, consisting of only one element.

Again, number of columns in Y = 1
= number of rows in X.

Thus we can also form the product YX as follows :

$$YX=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}[x_1 \quad x_2 \quad x_3]=\begin{bmatrix} y_1x_1 & y_1x_2 & y_1x_3 \\ y_2x_1 & y_2x_2 & y_2x_3 \\ y_3x_1 & y_3x_2 & y_3x_3 \end{bmatrix},$$

which is a matrix of order 3×3.

Example 2. Multiply the following matrices A, B to form the pooduct AB.

$$A=\begin{bmatrix} 2 & -3 \\ 7 & 1 \\ 0 & 5 \end{bmatrix},\quad B=\begin{bmatrix} 1 & 4 & 6 & -1 \\ \frac{1}{2} & 3 & -2 & 5 \end{bmatrix}.$$

Number of columns in A = 2
= number of rows in B.

Thus we can form the product AB as follows :

$$AB=\begin{bmatrix} 2(1)+(-3)(\tfrac{1}{2}) & 2(4)+(-3)(3) & 2(6)+(-3)(-2) & 2(-1)+(-3)(5) \\ 7(1)+1(\tfrac{1}{2}) & 7(4)+1(3) & 7(6)+1(-2) & 7(-1)+1(5) \\ 0(1)+5(\tfrac{1}{2}) & 0(4)+5(3) & 0(6)+5(-2) & 0(-1)+5(5) \end{bmatrix}$$

$$=\begin{bmatrix} \tfrac{1}{2} & -1 & 18 & -17 \\ 7\tfrac{1}{2} & 31 & 40 & -2 \\ 5/2 & 15 & -10 & 25 \end{bmatrix}.$$

AB is a matrix of order 3 × 4

Since number of columns in B $\neq$ number of rows in A, it is not possible to form the product BA in this case.

From the above examples it is clear that if A and B are any two matrices then

AB$\neq$BA (in general), that is the matrix multiplication is not commutative in general, although in some particular case AB may turn out to be equal to BA.

If A = B = C are any three matrices such that order of A is $m \times n$, order of B is $n \times p$ and order of C is $p \times q$, then we can form the product (AB) C by first forming the produc AB and then mlutiplying AB by C.

The matrix (AB) C will be of order $m \times q$.

It can also be proved that

$$(AB)\,C = A\,(BC)$$

The proof is beyond the scope of this book.

Exercise 1.1

1. Write the following matrices in tabular form.

(*i*) $A=[a_{ij}]$; $i=1, 2$ and $j=1, 2, 3$

(*ii*) $B=[b_{jk}]$; $j=1, 2$ and $k=1, 2, 3, 4$.

(*iii*) $C=[\Upsilon_{ik}]$; order 3×3.

(*iv*) $X=[x_{lm}]$; order 4×3.

(*v*) $T=[\lambda_{lk}]$; $l=1, 2, 3, 4$ and $k=1, 2, 3, 4$.

2. Which of the following matrices are conformable for addition ;

$$A=\begin{bmatrix} 2 & 3 & -4 \\ 0 & 1 & 0 \\ 3 & -5 & 6 \end{bmatrix} ; \; B=[b_{ij}] \text{ of order } 3\times3 ;$$

$C=[\mu_{lk}]$, $l=1, 2, 3$ and $k=1, 2, 3, 4$;

$$D=\begin{bmatrix} 1 & 1 & 1 & 5 \\ -\frac{1}{2} & 7 & -9 & 0 \\ \sqrt{3} & 6 & 2 & -4 \end{bmatrix} ; \; E=[\lambda_{ik}] \; i=1, 2, 3 \text{ and } k=1, 2, 3, 4.$$

$$X=\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \text{ and } Y=\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ d & c & b & a \end{bmatrix}$$

Indicate also the matrices that are not conformable for addition.

3. If $X=\begin{bmatrix} 4 & 3 & -2 & 1 \\ \sqrt{2} & 0 & 5 & \frac{1}{3} \\ \frac{2}{3} & \sqrt{5} & 1 & 1 \end{bmatrix}$; $Y=[\beta_{ik}]$, $i=1, 2, 3$ and $k=1, 2, 3, 4$

$L=\begin{bmatrix} a & h & g & d \\ h & b & f & d \\ g & f & c & d \end{bmatrix}$; $M=[\lambda_{ij}]$, order 3×3 and

$N=[\mu_{ik}]$, order 3×3.

Then compute the following .

(*i*) $X+Y$ (*ii*) $X+L$ (*iii*) $Y+L$

(*iv*) $Y+X$ (*v*) $L+X$ (*vi*) $(X+Y)+L$

(*vii*) $X+(Y+L)$ (*viii*) $L+M$ (*ix*) $M+N$

Also show that :

$$(X+Y)+L=X+(Y+L).$$

4. If $A=\begin{bmatrix} 2 & 3 & 0 & 5 \\ 1 & 6 & 4 & 7 \end{bmatrix}$, then compute

$A+A$, $(A+A)+A$. Also show that

(*i*) $2A+3A=5A$,

(*ii*) $3A-2A=A$ and

(*iii*) $A-A=O$, where O is a null matrix.

What is the order of O.

5. Which of the following matrices are conformable for multiplication ;

$A=\begin{bmatrix} 4 & 5 & -3 \\ 2 & 6 & 1 \\ 0 & 7 & 9 \end{bmatrix}$; $B=[b_{ij}]$, $i=1, 2, 3$ and $j=1, 2$;

$$C=\begin{bmatrix}2 & 3 & 1 & 5\\ 0 & 0 & 1 & 1\\ 2 & 2 & 2 & 2\end{bmatrix};\quad D=\begin{bmatrix}1 & 5 & 7 & 6\\ 3 & 3 & 3 & 3\\ 4 & 4 & 4 & 4\\ 1 & 1 & 1 & 1\end{bmatrix}$$

and $E=[e_{ik}]$, order 4×4.

6. If $A=\begin{pmatrix}2 & 3\\ 1 & 5\end{pmatrix}$; $B=\begin{pmatrix}1 & 7\\ 6 & 4\end{pmatrix}$; $C=\begin{pmatrix}2 & 3 & 1\\ 1 & 1 & 1\end{pmatrix}$;

$D=[d_{ij}]$; $i, j=1, 2, 3$; $E=\begin{bmatrix}e_{11} & e_{12} & e_{13}\\ e_{21} & e_{22} & e_{23}\\ e_{31} & e_{32} & e_{33}\end{bmatrix}$ and

$$F=\begin{bmatrix}-2 & -3\\ -4 & 1\\ 0 & 3\end{bmatrix}$$

Then compute the following :

(*i*) AB (*ii*) BA (*iii*) AC (*iv*) BC (*v*) CE
(*vi*) DE (*vii*) ED (*viii*) DF (*ix*) EF (*x*) (AB)C
(*xi*) (DE)F

Also answer the following questions as true or false by actually forming the products :

(*a*) AB = BA ?

(*b*) (AB)C = A (BC) ?

(*c*) (DE)F = D(EF) ?

Hint. To compute (AB)C, first compute AB and then compute (AB) C *i.e.*, post multiply AB by C.

7. If $A=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$ and $B=\begin{bmatrix}-3 & -2\\ 1 & -5\\ 4 & 3\end{bmatrix}$,

Find $X=\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\\ x_{31} & x_{32}\end{bmatrix}$ such that $A+B-X=O$.

8. If $A=\begin{bmatrix}1 & -1 & 1\\ 0 & 1 & 5\\ 2 & 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}2 & 4 & 5\\ 7 & 0 & 0\\ -2 & 3 & 1\end{bmatrix}$,

find $Y=\begin{bmatrix}y_1 & y_2 & y_3\\ y_4 & y_5 & y_6\\ y_7 & y_8 & y_9\end{bmatrix}$ Such that $Y=A-B$.

9. **Show that**

$$\begin{bmatrix}-1 & 2 & 3\\ 2 & 1 & 0\\ 3 & 5 & -1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}-a+2b+3c\\ 2a+b\\ 3a+5b-c\end{bmatrix}$$

10. Prove that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11}x+a_{12}y+a_{13}z \\ a_{21}x+a_{22}y+a_{23}z \\ a_{31}x+a_{32}y+a_{33}z \end{bmatrix}$$

11. If $A=[a_{ij}]$, $B=[b_{ij}]$, $C=[c_{ij}]$ are matrices of order 3×3, then show that :

(*i*) $A(B+C)=AB+AC$

(*ii*) $A+(B+C)=(A+B)+C$

(*iii*) $(AB)C=A(BC)$

12. If A and B are matrices, then explain why in general.

(*i*) $(A+B)^2 \neq A^2+2AB+B^2$

(*ii*) $(A-B)^2 \neq A^2-2AB+B^2$ and

(*iii*) $(A+B)(A-B) \neq A^2-B^2$

13. Verify that

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

14. Show that

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ANSWERS

1. (*i*) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ (*ii*) $\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix}$

(*iii*) $\begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$ (*iv*) $\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} \end{bmatrix}$

(*v*) $\begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{bmatrix}$

2. **A and B are conformable ;**

C, D, E and Y are conformable.

X is not conformable.

3. (*i*) $\begin{bmatrix} 4+\beta_{11} & 3+\beta_{12} & -2+\beta_{13} & 1+\beta_{14} \\ \sqrt{2}+\beta_{21} & \beta_{22} & 5+\beta_{23} & \frac{1}{3}+\beta_{24} \\ \frac{2}{3}+\beta_{31} & \sqrt{5}+\beta_{32} & 1+\beta_{33} & 1+\beta_{34} \end{bmatrix}$

(*ii*) $\begin{bmatrix} 4+a & 3+h & -2+g & 1+d \\ \sqrt{2}+h & 0+b & 5+f & \frac{1}{3}+d \\ \frac{2}{3}+g & \sqrt{5}+f & 1+c & 1+d \end{bmatrix}$

(*iii*) $$\begin{bmatrix} \beta_{11}+a & \beta_{12}+h & \beta_{13}+g & \beta_{14}+d \\ \beta_{21}+h & \beta_{22}+b & \beta_{2}+f & \beta_{24}+d \\ \beta_{31}+g & \beta_{32}+f & \beta_{33}+c & \beta_{34}+d \end{bmatrix}$$

(*iv*) $$\begin{bmatrix} \beta_{11}+4 & \beta_{12}+3 & \beta_{13}-2 & \beta_{14}+1 \\ \beta_{21}+\sqrt{2} & \beta_{22} & \beta_{23}+5 & \beta_{24}+\frac{1}{3} \\ \beta_{31}+\frac{2}{3} & \beta_{32}+\sqrt{5} & \beta_{33}+1 & \beta_{34}+1 \end{bmatrix}$$

(*v*) $$\begin{bmatrix} a+4 & h+3 & g-2 & d+1 \\ h+\sqrt{2} & b & f+5 & d+\frac{1}{3} \\ g+\frac{2}{3} & f+\sqrt{5} & c+1 & d+1 \end{bmatrix}$$

(*vi*) and (*vii*)

$$(X+Y)+L=X+(Y+L)=$$

$$\begin{bmatrix} 4+\beta_{11}+a & 3+\beta_{12}+h & -2+\beta_{13}+g & 1+\beta_{14}+d \\ \sqrt{2}+\beta_{21}+h & \beta_{22}+b & 5+\beta_{23}+f & \frac{1}{3}+\beta_{24}+d \\ \frac{2}{3}+\beta_{31}+g & \sqrt{5}+\beta_{32}+f & 1+\beta_{33}+c & 1+\beta_{34}+d \end{bmatrix}$$

(*viii*) L and M are not conformable for addition.

(*ix*) $$\begin{bmatrix} \gamma_{11}+\mu_{11} & \gamma_{12}+\mu_{12} & \gamma_{13}+\mu_{13} \\ \gamma_{21}+\mu_{21} & \gamma_{22}+\mu_{22} & \gamma_{23}+\mu_{23} \\ \gamma_{31}+\mu_{31} & \gamma_{32}+\mu_{32} & \gamma_{33}+\mu_{33} \end{bmatrix}$$

4. $$\begin{bmatrix} 4 & 6 & 0 & 10 \\ 2 & 12 & 8 & 14 \end{bmatrix}, \quad \begin{bmatrix} 6 & 9 & 0 & 15 \\ 3 & 18 & 12 & 21 \end{bmatrix},$$

O is of order 2×4.

5. A is conformable for products AB and AC ;
B is not conformable with any of them ;

C is conformable for products CD and CE ;
D is conformable for products DE and ED.

6.(*i*) $\begin{bmatrix} 20 & 26 \\ 31 & 27 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 9 & 38 \\ 16 & 38 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 7 & 9 & 5 \\ 7 & 8 & 6 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 9 & 10 & 8 \\ 16 & 22 & 10 \end{bmatrix}$

(*v*)
$$\begin{bmatrix} 2e_{11}+3e_{21}+e_{31} & 2e_{12}+3e_{22}+e_{32} & 2e_{13}+3e_{23}+e_{33} \\ e_{11}+e_{21}+e_{31} & e_{12}+e_{22}+e_{32} & e_{13}+e_{23}+e_{33} \end{bmatrix}$$

(*vi*)
$$\begin{bmatrix} d_{11}e_{11}+d_{12}e_{21}+d_{13}e_{31} & d_{11}e_{12}+d_{12}e_{22}+d_{13}e_{32} & d_{11}e_{13}+d_{12}e_{23}+d_{13}e_{3} \\ d_{21}e_{11}+d_{22}e_{21}+d_{23}e_{31} & d_{21}e_{12}+d_{22}e_{22}+d_{23}e_{32} & d_{21}e_{13}+d_{22}e_{23}+d_{23}e_{3} \\ d_{31}e_{11}+d_{32}e_{21}+d_{33}e_{31} & d_{31}e_{12}+d_{32}e_{22}+d_{33}e_{32} & d_{31}e_{13}+d_{32}e_{23}+d_{33}e_{3} \end{bmatrix}$$

(*vii*)
$$\begin{bmatrix} e_{11}d_{11}+e_{12}d_{21}+e_{13}d_{31} & e_{11}d_{12}+e_{12}d_{22}+e_{13}d_{32} & e_{11}d_{13}+e_{12}d_{23}+e_{13}d \\ e_{21}d_{11}+e_{22}d_{21}+e_{23}d_{31} & e_{21}d_{12}+e_{22}d_{22}+e_{23}d_{32} & e_{21}d_{13}+e_{22}d_{23}+e_{23}d_{3} \\ e_{31}d_{11}+e_{32}d_{21}+e_{33}d_{31} & e_{31}d_{12}+e_{32}d_{22}+e_{33}d_{32} & e_{31}d_{13}+e_{32}d_{23}+e_{33}d_{3} \end{bmatrix}$$

(*viii*)
$$\begin{bmatrix} -2d_{11}-4d_{12} & -3d_{11}+d_{12}+3d_{13} \\ -2d_{21}-4d_{22} & -3d_{21}+d_{22}+3d_{23} \\ -2d_{31}-4d_{32} & -3d_{31}+d_{32}+3d_{33} \end{bmatrix}$$

(*ix*) $$\begin{bmatrix} -2e_{11}-4e_{12} & -3e_{11}+e_{12}+3e_{13} \\ -2e_{21}-4e_{22} & -3e_{21}+e_{22}+3e_{22} \\ -2e_{31}-4e_{32} & -3e_{31}+e_{32}+3e_{33} \end{bmatrix}$$

(*x*) $$\begin{bmatrix} 66 & 86 & 46 \\ 89 & 120 & 58 \end{bmatrix}$$

(*xi*) $$\begin{bmatrix} -2\lambda_{11}-4\lambda_{12} & -3\lambda_{11}+\lambda_{12}+3\lambda_{13} \\ -2\lambda_{21}-4\lambda_{22} & -3\lambda_{21}+\lambda_{22}+3\lambda_{23} \\ -2\lambda_{31}-4\lambda_{32} & -3\lambda_{31}+\lambda_{32}+3\lambda_{33} \end{bmatrix},$$

where λ_{ij} is the (i, j) th element of 6 (*vi*).

(*a*) False (*b*) True (*c*) True

7. $$\begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}$$

8. $$\begin{bmatrix} -1 & -5 & -4 \\ -7 & 1 & 5 \\ 4 & 0 & 3 \end{bmatrix}$$

12. Since AB$\neq$BA in general.

1.7. *Some Types of Matrices.*

Definition 1: A square matrix $A^* = [a_{ij}]$ is called a *diagonal matrix* if

$$a_{ij} = 0 \ \forall i \neq j.$$

and all $a_{ij} \neq 0$

Thus

$$\overset{*}{A} = \begin{bmatrix} a_{11} & 0 & 0 \ldots\ldots\ldots\ldots 0 & 0 \\ 0 & a_{22} & 0 \ldots\ldots\ldots\ldots 0 & 0 \\ 0 & 0 & a_{33} \ldots\ldots\ldots\ldots 0 & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots 0 & a_{nn} \end{bmatrix}$$

is a diagonal matrix of order n.

The entries $a_{11}, a_{22}, \ldots\ldots\ldots\ldots, a_{nn}$ are said to constitute the *leading diagonal*.

If in the matrix A* we put $a_{11} = a_{22} = \ldots\ldots = a_{nn} = k$ (say) then we get

$$A = \begin{bmatrix} k & 0 & 0 & \ldots\ldots\ldots\ldots 0 & 0 \\ 0 & k & 0 & \ldots\ldots\ldots\ldots 0 & 0 \\ 0 & 0 & k & \ldots\ldots\ldots\ldots 0 & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots\ldots\ldots 0 & k \end{bmatrix}$$

Such a matrix is called a *scalar matrix* of order n.

Furthermore if $k=1$, then

$$I_n=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \cdots \\ \cdots & & & & & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

is said to be a *unit matrix* of order n.

It is to be noted (from above) that

$$A=k\,I_n$$

It is obvious that all scalar and unit matrices are diagonal matrices.

Examples.

(1) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

are diagonal matrices of order 2, 3 and 4 respectively.

(2) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}$

are scalar matrices of order 2, 3 and 4 respectively.

$$(3)\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are unit matrices of order 2, 3 and 4 respectively.

Definition 2. If $A=[a_{ij}]$ is a matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. It is denoted by A

Clearly $A'=[a_{ji}]$.

Examples.

$$(i)\quad \text{If } A=\begin{bmatrix} 2 & 3 & 0 & 3 \\ 1 & 5 & 4 & 2 \\ -6 & 2 & 7 & 8 \end{bmatrix}, \text{ then } A'=\begin{bmatrix} 2 & 1 & -6 \\ 3 & 5 & 2 \\ 0 & 4 & 7 \\ 3 & 2 & 8 \end{bmatrix}$$

$$(ii)\quad \text{If } B=\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \end{bmatrix}, \text{ then } B'=\begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \\ \beta_{13} & \beta_{23} \end{bmatrix}$$

Definition 3. A square matrix A is called *symmetric* if $A=A$

Clearly if $A=[a_{ij}]$ is a symmetric matrix, then

$$[a_{ij}]=[a_{ji}]\ \forall\ i, j.$$

Examples.

$$\text{The matrices } \begin{bmatrix} 2 & 4 \\ 4 & -3 \end{bmatrix}, \quad \begin{bmatrix} 5 & -3 & 7 \\ -3 & 6 & 0 \\ 7 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \sqrt{2} & 4 & -5 \\ \sqrt{2} & -6 & 8 & 7 \\ 4 & 8 & 0 & 1 \\ -5 & 7 & 1 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

are symmetric matrices of order 2, 3, 4 and 3 respectively.

Exercise 1.2

1. Specify the type of each of the following matrices.

(*i*) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$ (*ii*) $\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$

(*iii*) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(*v*) $\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ (*vi*) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

2. Which of the following matrices are symmetric ?

(*i*) $\begin{bmatrix} 2 & 3 & 5 \\ 3 & 0 & 1 \\ 5 & 1 & 7 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 6 & 8 & -9 & 5 \\ 8 & 3 & -3 & 2 \\ -9 & -3 & \sqrt{2} & 1 \\ 5 & 2 & 1 & 0 \end{bmatrix}$

(*iii*) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ (*iv*) $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & -a_5 & a_6 \end{bmatrix}$

(*v*) $\begin{bmatrix} 6 & 7 & 8 \\ 7 & 9 & 10 \\ -8 & 10 & 11 \end{bmatrix}$

ANSWERS

1. (*i*) Diagonal. (*ii*) Scalar.
(*iii*) Scalar. (*iv*) Diagonal.
(*v*) Scalar. (*vi*) Unit matrix of order 5.

2. (*i*), (*ii*) and (*iii*) are symmetric.

1.8. *Determinants*

Suppose that we have to solve the simultaneous linear equations,

$$a_1x_1+a_2x_2=c_1$$
$$b_1x_1+b_2x_2=c_2$$

we know that these equations involve two unknowns, *viz.*, x_1 and x_2, and their solution is given by

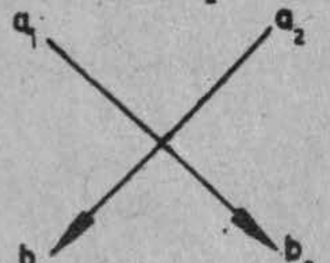
$$x_1=\frac{b_2c_1-a_2c_2}{a_1b_2-a_2b_1} \text{ and } x_2=\frac{c_2a_1-c_1b_1}{a_1b_2-a_2b_1}$$

provided $a_1b_2-a_2b_1\neq 0$.

The denominator $a_1b_2-a_2b_1$; which is same for x_1 and x_2, arises from the method of cross-multiplication. For example the coefficients of x_1 and x_2 in the above equations (written in the same order) ore

$$\begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix}$$

By cross-multiplication we see that



yields $a_1b_2-a_2b_1$

The quantity $a_1b_2-a_2b_1$ is called a *determinant of second order* and is written as :

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

clearly this determinant has two rows and two columns.

Since the coefficients of the equations are real numbers, the quantity $a_1b_2-a_2b_1$ is also a real number. Hence we note that the determinant

$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is a real number.

Solving in the same way the equations.

$$a_1x_1+a_2x_2+a_3x_3=h_1$$
$$b_1x_1+b_2x_2+b_3x_3=h_2$$
$$c_1x_1+c_2x_2+c_3x_3=h_3$$

we can see that the common denominator in the values of x_1, x_2, x_3 is a six termed expression given by :

$$a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1 \quad \ldots\ldots(i)$$

This is called a *determinant of order* 3 ! and is written as

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Rul› to expand a determinant of order 3.

Step. 1. We start from the term a_1, which is known as the leading term ;

now suppress the row and the column in which a_1 lies, we get

$$\begin{matrix} b_2 & b_3 \\ c_2 & c_3 \end{matrix}$$

, and by cross-multiplication we have the expression

$$a_1(b_2c_3-b_3c_2) \quad \ldots(1)$$

Step. 2. Take a_2 and suppress the row and the column in which a_2 lies to get the expression $a_2(b_1c_3-b_3c_1)$. Changing its sign we have the term

$$-a_2(b_1c_3-b_3c_1) \quad \ldots(2)$$

Step. 3. Finally take a_3 and suppress the row and the column in which a_3 lies to get the expression

$$a_3(b_1c_2-b_2c_1) \quad \ldots(3)$$

Adding expressions (1), (2) and (3) we get

$$a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$
$$=a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1,$$

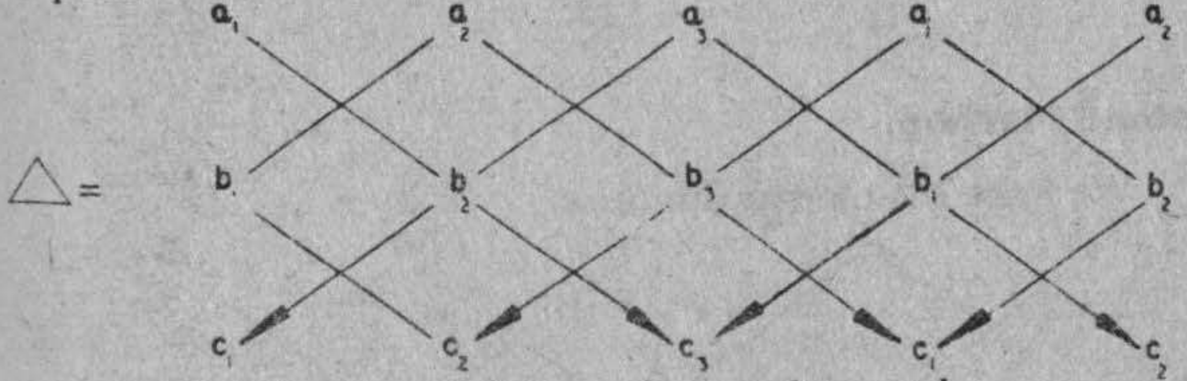
which is the value of the given determinant.

NOTE:—It should be carefully noted that in this rule the leading term is prefixed by +sign, the second by −sign and the third by +sign.

This rule is only a convenient way to express the determinant Δ in the form of an expression.

Alternate Method.

Another frequently used device for remembering the expansion of a third order determinant is given below :—



$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3,$$

which is same as (*i*) on page 36.

In the above device the result is obtained by adding the products of the triples in the respective diagonals shown by arrows pointing to the right and by subtracting the products of triples in the respective diagonals shown by arrows pointing to the left.

Example 1. Expand the determinant

$$\begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix}$$

The given determinant $= 2\times 7-4\times -3$

$= 14+12=26$

Example 2. Expand the determinant

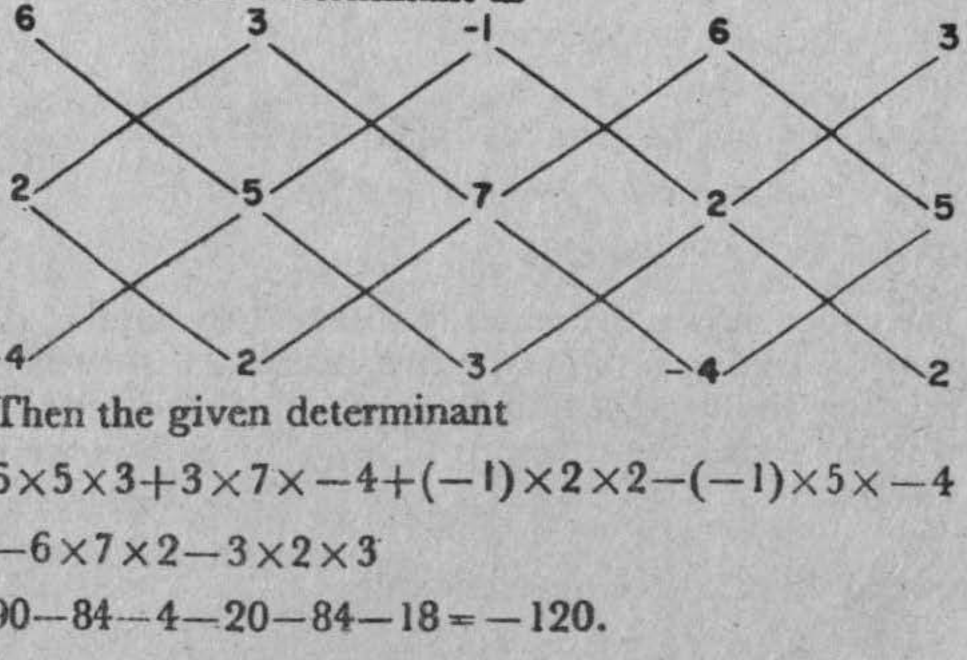
$$\begin{vmatrix} 6 & 3 & -1 \\ 2 & 5 & 7 \\ -4. & 2 & 3 \end{vmatrix}$$

First method

$$\begin{aligned}\text{The given determinant} &= 6(5\times3-2\times7)-3(2\times3\\ &\quad -(-4)\times7)+(-1)(2\times2-(-4)\times5)\\ &= 6(15-14)-3(6+28)-1(4+20)\\ &= 6(1)-3(34)-1(24)=6-102-24\\ &= -120.\end{aligned}$$

Second Method.

We write the determinant as



Then the given determinant

$$=6\times5\times3+3\times7\times-4+(-1)\times2\times2-(-1)\times5\times-4$$

$$-6\times7\times2-3\times2\times3$$

$$=90-84-4-20-84-18=-120.$$

1.9. *Properties of Determinants.*

1.9.1. *The value of a determinant is unaltered by interchanging its rows and columns. i.e.*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We have $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2\,c_3 - b_3\,c_2) - b_1(a_2\,c_3 - a_3\,c_2) + c_1(a_2\,b_3 - a_3\,b_2)$

$$= a_1\,b_2\,c_3 - a_1\,b_3\,c_2 + a_2\,b_3\,c_1 - a_2\,b_1\,c_3 + a_3\,b_1\,c_2 - a_3\,b_2\,c_1$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

1 9. 2. *The interchange of two rows, or of two columns, changes the sign of a determinant without altering its numerical value.*

For example $\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = b_1(a_2c_3 - a_3c_2) - b_2(a_1c_3 - a_3c_1) + b_3(a_1c_2 - a_2c_1)$

$$= -(a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1).$$

$$= - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

1.9. 3 *If two rows or two columns of a determinant are identical the value of the determinant is zero.*

We have $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(a_2c_3 - a_3c_2) - a_2(a_1c_3 - a_3c_1) + a_3(a_1c_2 - a_2c_1).$

$$= a_1a_2c_3 - a_1a_2c_3 - a_1a_3c_2 + a_1a_3c_2 + a_2a_3c_1 - a_2a_3c$$
$$= 0.$$

Similarly it can be shown that

$$\begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \ldots\ldots\ldots\ldots = 0$$

1.9. 4. *If all the entries of a row (or column) of a determinant are zeros then the value of the determinant is zero.*

For example $\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(0-0)-a_2(0-0)+a_3(0-0)$
$= 0$

1.9. 5. *If every element in a row, or column, of a determinant is multiplied by a number k, then the determinant is multiplied by k.*

For example $\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= ka_1(b_2c_3-b_3c_2)-ka_2(b_1c_3-b_3c_1)+ka_3(b_1c_2-b_2c_1)$$
$$= k[a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1]$$
$$= k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_1 & c_3 \end{vmatrix}$$

Similarly it can be shown that

$$k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ kb_1 & kb_2 & kb_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ kc_1 & kc_2 & kc_3 \end{vmatrix}$$

$$= \begin{vmatrix} ka_1 & a_2 & a_3 \\ kb_1 & b_2 & b_3 \\ kc_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & ka_2 & a_3 \\ b_1 & kb_2 & b_3 \\ c_1 & kc_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & ka_3 \\ b_1 & b_2 & kb_3 \\ c_1 & c_2 & kc_3 \end{vmatrix}$$

1.9. 6. The determinant

$\begin{vmatrix} a_1+\alpha_1 & a_2 & a_3 \\ b_1+\beta_1 & b_2 & b_3 \\ c_1+\gamma_1 & c_2 & c_3 \end{vmatrix}$ is equal to the sum of the

determinants $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\begin{vmatrix} \alpha_1 & a_2 & a_3 \\ \beta_1 & b_2 & b_3 \\ \gamma_1 & c_2 & c_3 \end{vmatrix}$

Hint. To prove that

$$\begin{vmatrix} a_1+\alpha_1 & a_2 & a_3 \\ b_1+\beta_1 & b_2 & b_3 \\ c_1+\gamma_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & a_2 & a_3 \\ \beta_1 & b_2 & b_3 \\ \gamma_1 & c_2 & c_3 \end{vmatrix}$$

just expand the determinants on the left hand side by the first column.

$$\begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3 \\ b_1+\beta_1 & b_2+\beta_2 & b_3 \\ c_1+\gamma_1 & c_2+\gamma_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_1 & a_2 & a_3 \\ \beta_1 & b_2 & b_3 \\ \gamma_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \alpha_2 & a_3 \\ b_1 & \beta_2 & b_3 \\ c_1 & \gamma_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & a_3 \\ \beta_1 & \beta_2 & b_3 \\ \gamma_1 & \gamma_2 & c_3 \end{vmatrix}$$

and that

$$\begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 \\ b_1+\beta_1 & b_2+\beta_2 & b_3+\beta_3 \\ c_1+\gamma_1 & c_2+\gamma_2 & c_3+\gamma_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \cdots\cdots + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix},$$

where L.H.S. is the sum of 8 determinants.

1.9.7 *The value of a determinant is unaltered if to each element of a row (or column) is added a constant multiple of the corresponding element of another row (or column.)*

For example
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\lambda a_2 & a_2 & a_3 \\ b_1+\lambda b_2 & b_2 & b_3 \\ c_1+\lambda c_2 & c_2 & c_3 \end{vmatrix}$$

The proof is simple, since

$$\text{R.H.S.} = \begin{vmatrix} a_1+\lambda a_2 & a_2 & a_3 \\ b_1+\lambda b_2 & b_2 & b_3 \\ c_1+\lambda c_2 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda a_2 & a_2 & a_3 \\ \lambda b_2 & b_2 & b_3 \\ \lambda c_2 & c_2 & c_3 \end{vmatrix} \text{(By 1.9.6)}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \lambda \begin{vmatrix} a_2 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix} \text{(By 1.9.5)}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 0 \quad \text{(By 1.9.4)}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{L.H.S.}$$

Similarly it can be shown that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\lambda b_1+\mu c_1 & b_1+\nu c_1 & c_1 \\ a_2+\lambda b_2+\mu c_2 & b_2+\nu c_2 & c_2 \\ a_3+\lambda b_2+\mu c_3 & b_3+\nu c_3 & c_3 \end{vmatrix}$$

Exercise 1.3.

1. Expand the following determinants :

(i) $\begin{vmatrix} 2 & 3 \\ 7 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 5 & \sqrt{3} \\ 6 & -2 \end{vmatrix}$ (iii) $\begin{vmatrix} -2 & 3 & -1 \\ 4 & 5 & 2 \\ 1 & 7 & 0 \end{vmatrix}$

(iv) $\begin{vmatrix} 1 & 5 & -4 \\ -2 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix}$ (v) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

2. Verify that

(*i*) $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$ (*ii*) $$\begin{vmatrix} m+1 & m+2 & m+3 \\ m+4 & m+5 & m+6 \\ m+7 & m+8 & m+9 \end{vmatrix} = 0$$

(*iii*) $$\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

(*iv*) $$\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$$

(*v*) $$\begin{vmatrix} \lambda^2 & \lambda & 1 \\ \mu^2 & \mu & 1 \\ \nu^2 & \nu & 1 \end{vmatrix} = -(\lambda-\mu)(\mu-\nu)(\nu-\lambda)$$

3. Without expansion verify that

(*i*) $$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$ (*ii*) $$\begin{vmatrix} 2 & -3 & 6 \\ 4 & -7 & 12 \\ 6 & 5 & 18 \end{vmatrix} = 0$$

(*iii*) $$\begin{vmatrix} 10 & 3 & -5 \\ 7 & 8 & -2 \\ -30 & -9 & 15 \end{vmatrix} = 0$$ (*iv*) $$\begin{vmatrix} bc & ca & ab \\ a^3 & b^3 & c^3 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix} = 0$$

(*v*) $$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}=0$$

(*vi*) $$\begin{vmatrix} 10 & 4 & 1 \\ 9 & -5 & -3 \\ -6 & 7 & 2 \end{vmatrix}=\begin{vmatrix} 10 & 3 & 1 \\ 9 & 1 & -3 \\ -6 & 10 & 2 \end{vmatrix}+\begin{vmatrix} 10 & 1 & 1 \\ 9 & -6 & -3 \\ -6 & -3 & 2 \end{vmatrix}$$

(*vii*) $$\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}=\begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

ANSWERS

1. (*i*) -15. (*ii*) $-10-6\sqrt{3}$. (*iii*) 11. (*iv*) 98.
   (*v*) $abc+2fgh-af^2-bg^2-ch^2$

1.10. *Linear Transformation*

The set of equations

$$\begin{aligned} y_1&=a_{11}x_1+a_{12}x_2 \\ y_2&=a_{21}x_1+a_{22}x_2\,; \end{aligned} \qquad \text{......(1)}$$

where a_{11}, a_{12}, a_{21}, a_{22} are given constants and x_1, x_2, y_1, y_2 are variables, is said to be a *linear transformation in two dimensions*, connecting the variables x_1, x_2 and the variables

y_1, y_2. When $a_{11}, a_{12}, a_{21}, a_{22}$ are real numbers we say that the transformation is real.

The equations (1) can be written in the matrix form as

$$Y = AX,$$

where $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

The matrix A is called *the matrix of this linear transformation*.

The determinant

$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is called the modulus of the linear transformation (1).

It is to be noted that we shall denote by $|A|$ the determinant of the matrix A.

Example. If $y_1 = 2x_1 + 3x_2$

$y_2 = 4x_1 - 5x_2$.

Find the matrix and the modulus of the linear transformation.

The matrix of the linear transformation is given by

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix},$$

and the modulus is

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = -10 - 12 = -22.$$

1.11. *Adjoint of a Matrix.* Suppose that we are given a square matrix

$$A=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

The matrix $\begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}$

is said to be the adjoint of the matrix **A**. It is denoted by Adj A.

Examples. The adjoints of the matrices

$X=\begin{bmatrix} 1 & 7 \\ 3 & 5 \end{bmatrix}$ and $Y=\begin{pmatrix} 0 & -6 \\ \sqrt{2} & -4 \end{pmatrix}$ are given by

$\text{Adj } X=\begin{bmatrix} 5 & -7 \\ -3 & 1 \end{bmatrix}$ and $\text{Adj } Y=\begin{pmatrix} -4 & 6 \\ -\sqrt{2} & 0 \end{pmatrix}$

respectively.

From above we have

$$X\,(\text{Adj } X)=\begin{pmatrix} 1 & 7 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 5 & -7 \\ -3 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 5-21 & -7+7 \\ 15-15 & -21+5 \end{pmatrix}$$

$$=\begin{pmatrix} -16 & 0 \\ 0 & -16 \end{pmatrix}=-16\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \ldots\ldots\ldots(1)$$

But $|X|=\begin{pmatrix} 1 & 7 \\ 3 & 5 \end{pmatrix}=5-21=-16.$

Therefore we have from (1)

$X\,(\mathrm{Adj}\ X) = |X|\,I_2$, where I_2 is a unit matrix of order 2.

$$\text{Again}\quad (\mathrm{Adj}\ X)\,X = \begin{pmatrix} 5 & -7 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5-21 & 35-35 \\ -3+3 & -21+5 \end{pmatrix}$$

$$= -16 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |X|\,I_2.$$

Hence $\quad X\,(\mathrm{Adj}\ X) = (\mathrm{Adj}\ X)\,X = |X|\,I_2$.

Similarly we can prove that

$$Y\,(\mathrm{Adj}\ Y) = (\mathrm{Adj}\ Y)\,Y = |Y|\,I_2$$

It may be noted that if A is any square matrix of order n, then

$A\,(\mathrm{Adj}\ A) = (\mathrm{Adj}\ A)\,A = |A|\,I_n$, where I_n is a unit matrix of order n.

Definition. A square matrix X is said to be *singular* if $|X| = 0$.

If $|X| \neq 0$, the matrix X is called *non-singular*,

Examples.

(*i*) The matrix $X = \begin{pmatrix} 3 & 5 \\ 6 & 10 \end{pmatrix}$ is singular, since

$$|X| = \begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix} = 30 - 30 = 0.$$

(*ii*) The matrix $A=\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 4 \\ 6 & -2 & 1 \end{bmatrix}$ is singular,

$$\text{since } |A| = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 1 & 4 \\ 6 & -2 & 1 \end{vmatrix} \begin{aligned} &=2(1+8)+3(-6) \\ &=18-18=0. \end{aligned}$$

(*iii*) The matrix $M=\begin{pmatrix} -3 & 2 \\ 4 & 7 \end{pmatrix}$ is non-singular,

$$\text{since } |M| = \begin{vmatrix} -3 & 2 \\ 4 & 7 \end{vmatrix} = -21-8=-29 \neq 0.$$

(*iv*) The matrix $I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is non-singular

$$\text{since} \qquad |I_3| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

It may be noted that all unit matrices are essentially non-singular.

1.12. *Inverse of a Matrix.* If A is a non-singular square matrix, then the matrix $\frac{\text{Adj A}}{|A|}$ is said to be the inverse of the matrix A. It is denoted by A^{-1}.

i.e. $A^{-1} = \frac{\text{Adj A}}{|A|}$

Example. Find the inverse of the matrix

$$\begin{bmatrix} 3 & 1 \\ -4 & 6 \end{bmatrix}$$

Let $$A = \begin{bmatrix} 3 & 1 \\ -4 & 6 \end{bmatrix}$$

Now $$|A| = \begin{vmatrix} 3 & 1 \\ -4 & 6 \end{vmatrix} = 18+4=22(\neq 0),$$

so that the matrix A is non-singular.

Also $$\text{Adj } A = \begin{bmatrix} 6 & -1 \\ 4 & 3 \end{bmatrix}, \text{ thus}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \tfrac{1}{22}\begin{bmatrix} 6 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{6}{22} & -\frac{1}{22} \\ \frac{4}{22} & \frac{3}{22} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{11} & -\frac{1}{22} \\ \frac{2}{11} & \frac{3}{22} \end{bmatrix}$$

Clearly $$AA^{-1} = \begin{bmatrix} 3 & 1 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} \frac{3}{11} & -\frac{1}{22} \\ \frac{2}{11} & \frac{3}{22} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{11}+\frac{2}{11} & -\frac{3}{22}+\frac{3}{22} \\ -\frac{12}{11}+\frac{12}{11} & \frac{4}{22}+\frac{18}{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Similarly we may prove that

$$A^{-1}A = I_2.$$

It may be noted that if A is a non-singular square matrix of order n, then

$$AA^{-1} = A^{-1}A = I_n.$$

1.13. *Simultaneous Linear Equations.*

The theory of matrices has a wide range of applications in various branches of mathematics and physics. In this section we shall apply this theory to the solution of 2 non-homogeneous linear equations, in two unknowns, with constant coefficients. The theory can be extended to the solution of '*m*' homogeneous or non-homogeneous linear equations in '*n*' unknowns, but the discussion is beyond the scope of the present book.

Consider the equations

$$\begin{aligned} a_1 x_1 + a_2 x_2 &= b_1 \\ a_3 x_1 + a_4 x_2 &= b_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1) \end{aligned}$$

x_1, x_2 are variables and a_1, a_2, a_3, a_4, b_1, b_2 are constants.

Using the notation of matrices the equations (1) can be written as

$$AX = B,$$

where $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

If the matrix A is non-singular, the equations (1) have a *unique** solution.

To find the solution we proceed as follows :

Since A is non-singular A^{-1} exists and so the solution of the system of equations $AX = B$ is $X = A^{-1}B$ *i.e.*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{1}{|A|} \begin{bmatrix} a_4 b_1 - a_2 b_2 \\ -a_3 b_1 + a_1 b_2 \end{bmatrix}$$

---

*There are many cases which should have been discussed, but the discussion is beyond the scope of this book.

or $$x_1 = \frac{a_4 b_1 - a_2 b_2}{|\,A\,|} \text{ and } x_2 = \frac{a_1 b_2 - a_3 b_1}{|\,A\,|}$$

or $$x_1 = \frac{a_4 b_1 - a_2 b_2}{|\,A\,|} = \frac{\begin{pmatrix} b_1 & a_2 \\ b_2 & a_4 \end{pmatrix}}{|\,A\,|} \text{ and}$$

$$x_2 = \frac{a_1 b_2 - a_3 b_1}{|\,A\,|} = \frac{\begin{pmatrix} a_1 & b_1 \\ a_3 & b_2 \end{pmatrix}}{|\,A\,|}$$

This result is known as Cramer's rule, and according to this rule .

x_1 is a quotient of two determinants, whose denominator is | A | and whose numerator is obtained from | A | by substituting for the first column of | A | the column $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Also x_2 is a quotient of two determinants, whose denominator is | A | and whose numerator is obtained from | A | by substituting for the second column of | A | the column $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Example I.

Solve the equations

$$2x + 3y = 5$$

$$x - y = 7 \text{ uniquely.}$$

Here $|\,A\,| = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -2 - 3 = -5 (\neq 0).$

Thus the system of equations (1) has a unique solution only if $|\,A\,| \neq 0$.

Thus $$x=\frac{\begin{vmatrix} 5 & 3 \\ 7 & -1 \end{vmatrix}}{|A|}=\frac{-5-21}{-5}=\frac{26}{5}$$

and $$y=\frac{\begin{vmatrix} 2 & 5 \\ 1 & 7 \end{vmatrix}}{|A|}=\frac{14-5}{-5}=\frac{-9}{5}$$

is the unique solution of the given equations.

Example 2. Show that the solution set of the system of equations

$$\begin{aligned} 2x+3y&=5 \\ 4x+6y&=8 \end{aligned}$$

is an empty set.

Since $|A|=\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}=12-12=0$, the equations do not have a unique* solution, *i.e.* empty set.

Exercise 1.4

1. Write down the matrix of each of the following linear transformations.

(*i*) $$\begin{aligned} a_{11}x_1+a_{12}x_2&=b_1 \\ a_{21}x_1+a_{22}x_2&=b_2 \end{aligned}$$

(*ii*) $$\begin{aligned} ax+by&=c \\ fx+gy&=h \end{aligned}$$

---

*As a matter of fact, the equations have no solution at all in this case (*iii*) $$\begin{aligned} 4x-3y&=9\\ \sqrt{2}\,x+7y&=\tfrac{1}{2}\end{aligned}$$

(*iv*) $$\begin{aligned} lx+my&=n\\ 5x-10y&=11.\end{aligned}$$

2. Find the adjoint of each of the following matrices:

(*i*) $\begin{bmatrix} 4 & -7 \\ 9 & 11 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (*iv*) $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

(*v*) $\begin{bmatrix} \lambda & \mu \\ \alpha & \beta \end{bmatrix}$

3. Which of the following matrices are singular and which of them are non-singular.

(*i*) $\begin{bmatrix} 4 & 7 \\ 7 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & \sqrt{2} \\ 12 & \sqrt{72} \end{bmatrix}$

(*iii*) $\begin{bmatrix} 1 & 0 \\ 10 & 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

(*v*) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & e \end{bmatrix}$ (*vi*) $\begin{bmatrix} 1 & -1 & 6 \\ 2 & 4 & -2 \\ 3 & 5 & \frac{2}{3} \end{bmatrix}$

4. Find the value of λ such that the following matrices are singular.

(*i*) $\begin{bmatrix} \lambda & 2 \\ 3 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} \lambda & \lambda \\ 7 & -6 \end{bmatrix}$

(*iii*) $\begin{bmatrix} \lambda & 50 \\ 2 & \lambda \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & -3 & 5 \\ \frac{1}{2} & 4 & 6 \\ \lambda & 0 & 8 \end{bmatrix}$

(*v*) $\begin{bmatrix} 5 & 0 & 9 \\ 8 & \lambda & -8 \\ 2 & 2 & 4 \end{bmatrix}$

5. Find the inverse of the following matrices :

(*i*) $\begin{bmatrix} 4 & 9 \\ 7 & 6 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -\sqrt{3} & \frac{1}{2} \\ \frac{2}{3} & \sqrt{6} \end{bmatrix}$

(*iii*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (*iv*) $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

(*v*) $\begin{bmatrix} 2 & 3 \\ 4 & \lambda \end{bmatrix}$ (*vi*) $\begin{bmatrix} l & m \\ p & q \end{bmatrix}$

6. Solve (if possible) the following systems of equations by the Cramer's rule.

(*i*) $x_1+x_2=7$
$x_1-x_2=9$

(*ii*) $\frac{1}{2}x+7y=3$
$5x+11y=4$

(*iii*) $lx+my=n$
$px+qy=r$

(*iv*) $4x+9y=11$
$20x+45y=100$

(*v*) $\sqrt{3}\,x+12y=1$
$7\sqrt{3}\,x+84y=10$

(*vi*) $18x_1+9x_2=3$
$5x_1+3x_2=10$

Answers

1. (*i*) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (*ii*) $\begin{bmatrix} a & b \\ f & g \end{bmatrix}$

(*iii*) $\begin{bmatrix} 4 & -3 \\ \sqrt{2} & 7 \end{bmatrix}$ (*iv*) $\begin{bmatrix} l & m \\ 5 & -10 \end{bmatrix}$

2. (*i*) $\begin{bmatrix} 11 & 7 \\ -9 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (*iv*) $\begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$

(*v*) $\begin{bmatrix} \beta & -\mu \\ -\alpha & \lambda \end{bmatrix}$

3. (*i*), (*iv*), (*v*) and (*vi*) are non-singular, (*ii*) and (*iii*) are singular.

4. (*i*) $\frac{3}{2}$. (*ii*) 0. (*iii*) ± 10.

(*iv*) 2. (*v*) -112.

5. (*i*) $\begin{bmatrix} -\frac{2}{13} & \frac{3}{13} \\ \frac{7}{39} & -\frac{4}{39} \end{bmatrix}$

(*ii*) $\dfrac{-3}{3\sqrt{15}+1}\begin{bmatrix} \sqrt{5} & -\frac{1}{2} \\ -\frac{2}{3} & -\sqrt{3} \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(*iv*) $\dfrac{1}{\lambda\mu}\begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix}$; $\lambda, \mu \neq 0$

(*v*) $\dfrac{1}{2(\lambda-6)}\begin{bmatrix} \lambda & -3 \\ -4 & 2 \end{bmatrix}$; $\lambda \neq 6$.

(*vi*) $\dfrac{1}{lq-mp}\begin{bmatrix} q & -m \\ -p & l \end{bmatrix}$; $lq-mp \neq 0$..

6. (*i*) $x_1=8$, $x_2=-1$. (*ii*) $x=-\frac{10}{59}$, $y=\frac{26}{59}$.

(*iii*) $\dfrac{nq-mr}{lq-mp}$, $y=\dfrac{lr-np}{lq-mp}$.

(*iv*) **No solution.**

(*v*) **No solution.**

(*vi*) $x_1=-9$, $x_2=\frac{55}{3}$.

CHAPTER II

FUNCTIONS AND LIMITS

2.1. *Interval*. In the theory of calculus we are concerned with real functions whose *domains* are subsets of the x-axis. These subsets are actually intervals on the real line lying between two prescribed real numbers. These sets are called (some-times) *bounded intervals* and the prescribed real numbers are said to be the *end points*. sometimes it is important to distinguish between intervals which include their end points and those which do not. This distinction is made by introducing the following definitions :

If a, b are two real numbers such that $a<b$ we denote by $[a, b]$ the collection of all real numbers x satisfying the inequality $a \leqslant x \leqslant b$ and refer to this set as the *closed interval* a, b. The corresponding *open interoal*, written as $]a, b[$ is the set of all real numbers x satisfying $a < x < b$. It should be noted that the closed interval $[a, b]$ includes the end points a and b, whereas the open interval does not. (see figure 2.1)

The open interval $]a, b[$ is also called the *interior* of $[a, b]$. Half-open intervals $[a, b[$ and $]a, b]$ which inlcude just one end point, are defined by the conditions $a<x \leqslant b$ and $a \leqslant x<b$; respectively.

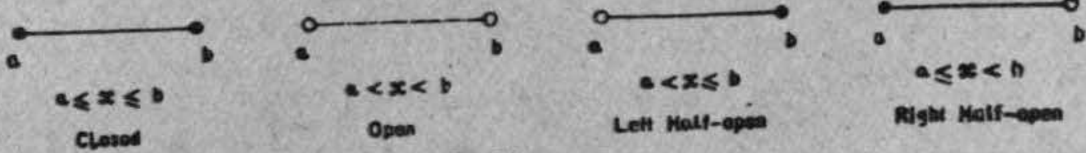


Fig. 2.1

Let f be a function whose domain is a closed interval $[a, b]$. The portion of the plane between the graph of f

and the x-axis is called the *ordinate set* of f. More precisely the ordinate set of f is the collection of all points (x, y) satisfying the conditions

$$a \leqslant x \leqslant b, \quad 0 \leqslant y \leqslant f$$

In each of the examples shown in figure 2.2, the shaded portion represents the ordinate set of the corresponding function.

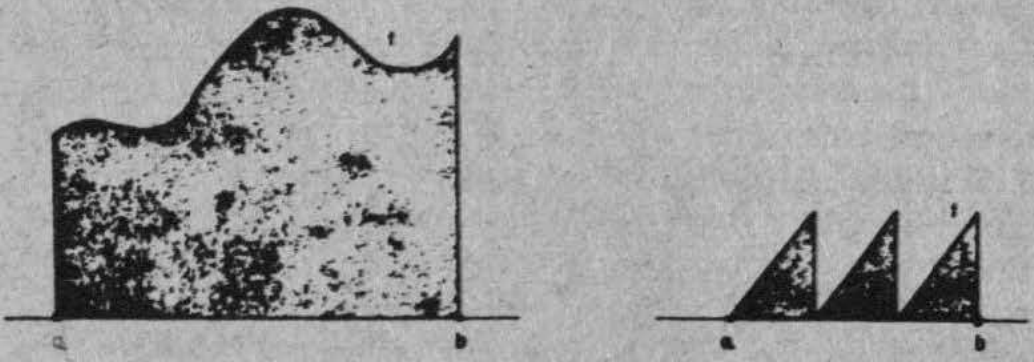


Fig. 2.2

Ordinate sets are geometric objects whose areas we want to compute by means of the integral calculus.

2.2. *Cartesian coordinate system for plane (2-space).* In this system two perpendicular reference lines (*called the coordinate axes*) are chosen, one horizontal (called the " x-axis "), the other vertical (the "y-axis"). Their point of intersection, denoted by O, is called the origin. On the x-axis a convenient point is chosen to the right of O, and its distance from O is called the unit distance. Vertical distances along the y-axis are measured with the same unit distance. Now each point in the plane (2-space) is assigned a pair of real numbers, called its *coordinates*. These coordinates represent the distances from the point to the axes. Figure 2.3. illustrates some examples. The point P with coordinates (3,2) lies three units to the right of the y-axis and two units above the x-axis. The number 3 is called the x-coordinate of the point, while 2 is its *y*-coordinate. Points to the left of *y*-axis have a negative *x*-coordinate, whereas points below the *x*-axis have a negative *y*-coordinate. The *x*-ccordinate of a point is usually called its *abscissa*, and the *y*-coordinate is known as its *ordinate*.

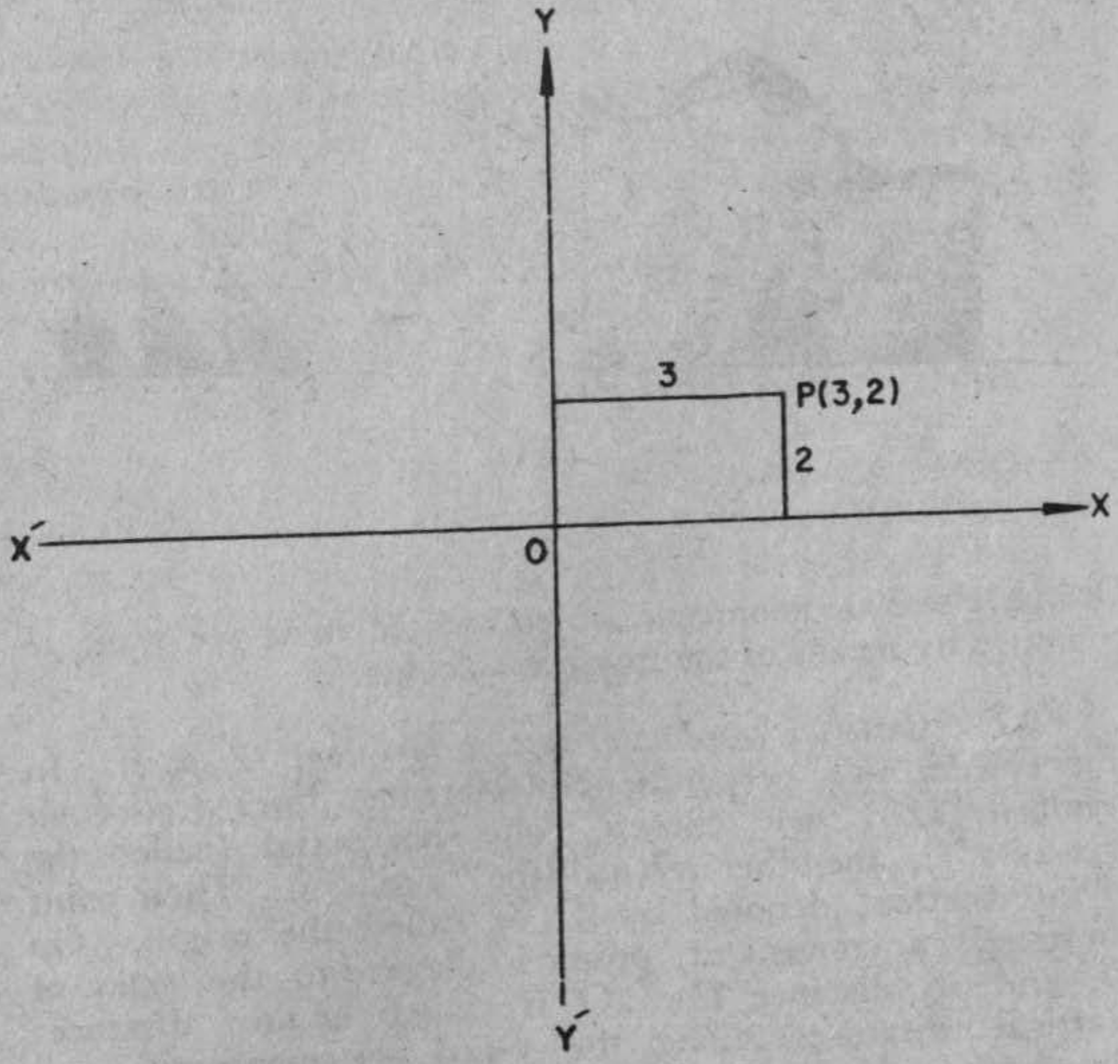


Fig 2.3

When we write a pair of real numbers (a, b) to represent a point, we agree that the abscissa a, is always written first. For this reason, the pair (a, b) is referred to as an *ordered pair*. It is clear that two ordered pairs (a, b) and (c, d) represent the same point if and only if $a=c$ and $b=d$. Point (a, b) with $a>0$, $b>0$ is said to lie in the *first quadrant* ; that with $a<0$, $b>0$ is in the *second quadrdnt* ; that with $a<0$, $b<0$ is in the *third quadrant* and that with $a>0$, $b<0$ lies in the *fourth qudrant*. It is to be noted that coordinates of the origin O are $(0, 0)$.

A geometric figure, such as a curve in a plane (2-space), is a collection of points satisfying cartain conditions. By translating these conditions into analytic expressions involving the coordinates x and y, we obtain one or more equations which characterize the curve. For example, suppose the curve is a circle of radius r with its centre at the origin (figure 2 4). Let P be an arbitrary point on this circle having (x, y) as its coordinates (such a point will be referred to as the point P (x, y). Then the line segment O P is the hypotenuse of a right triangle whose legs have lengths $[x|$ and $|y|$ and hence by Pathagoras' theorem,

$$x^2+y^2=r^2$$

This equation is said to be the cartesian equation of the circle and is satisfied by all points (x, y) on the circle and by no others. So this equation completely characterizes the circle. This example illustrates how analytic geometry is used to convert geometrical statements about points to analytical statements about real numbers.

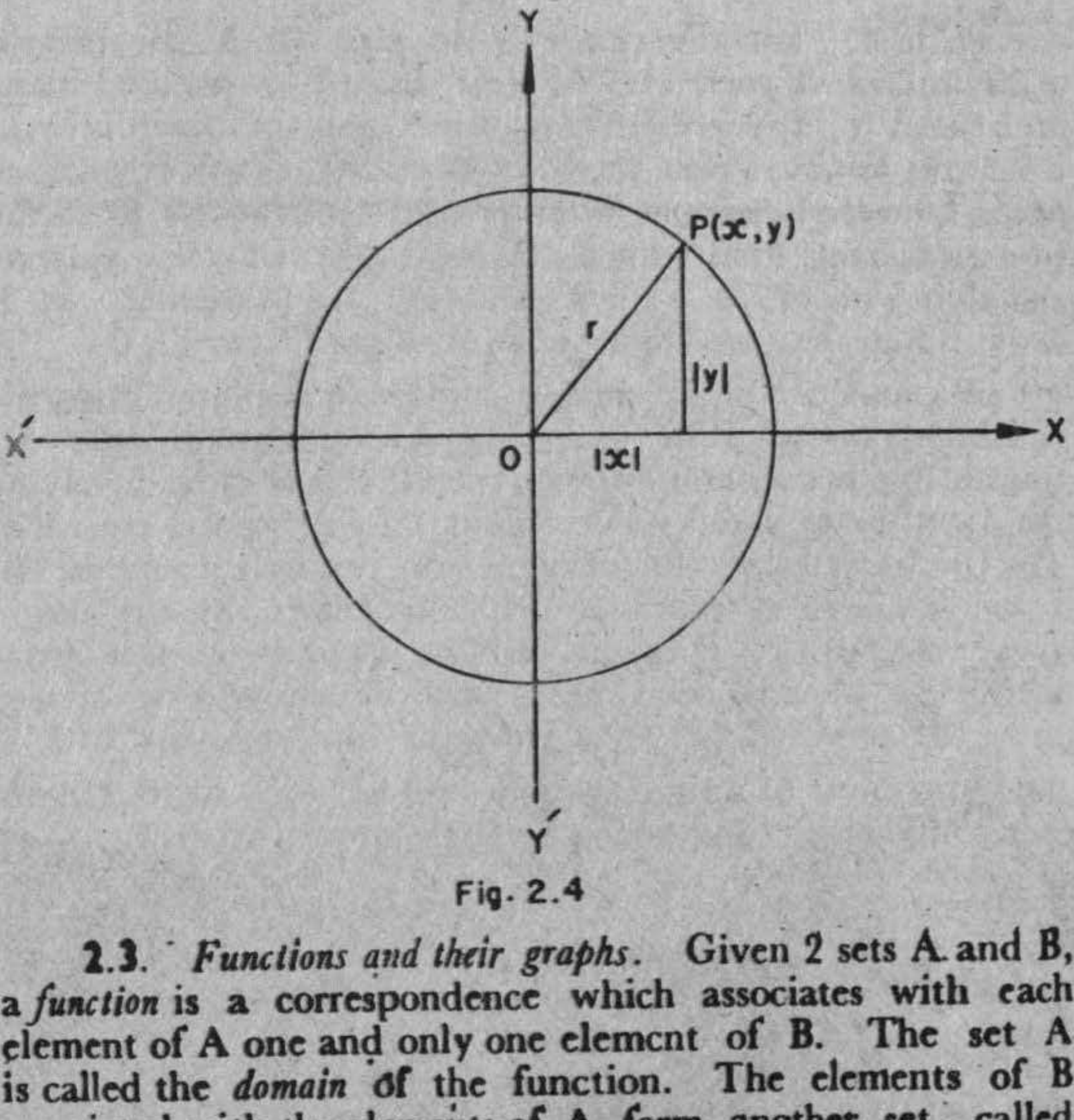


Fig. 2.4

2.3. *Functions and their graphs.* Given 2 sets A and B, a *function* is a correspondence which associates with each element of A one and only one element of B. The set A is called the *domain* of the function. The elements of B associated with the elements of A form another set, called the *range* of the function (This need not be the whole set B, but it is a subset of B).

In this book we shall usually use letters f, g, h and α, β, $\varnothing$, ψ to denote functions. If f is a given function and x is an element of its domain. the notation $f(x)$ is used to denote that element of the range which is associated with x by the function f. It is called the *image of* x *under* f or *the value of* f *at* x. The symbol $f\cdot(x)$ is read as "f of x. In calculus it is customary to write y for $f(x)$. The element (variable) x is said to be the *independent variable*, whereas the image y of x is said to be the *dependent variable*.

Example 1.

The area of a circle is a function of its radius since given any radius we can readily find area of the cirle associated with this radius. If r is the radius of given circle, then the area A is given by the formula $A = \pi r^2$. Here r is the independent variable and A the depondent variable.

Example 2.

The volume of a cube is a function of the length of its any edge. If the length of any edge is x, then the volume V is given by the formula $V = x^3$. Here x is the independent and V the dependent variable.

Example 3.

Draw the graph of the function $f(x) = x$.

Writing y for $f(x)$ we have $y = x$ for each point (x, y) on the graph. The graph is clearly a straight line making equal angles with the coordinate axes (figure 2.5). The domain and range of this function is the set R of all real numbers. This function is often called the *identity function.*

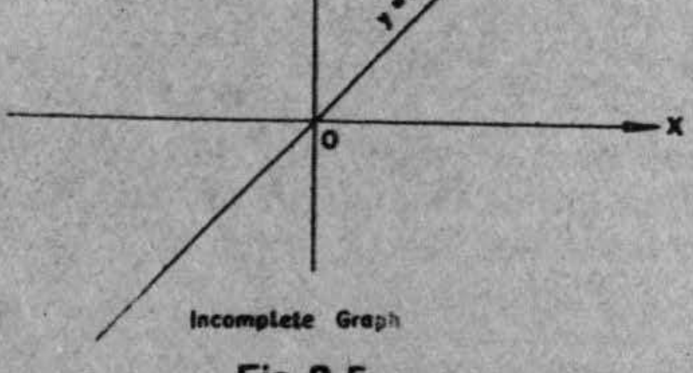
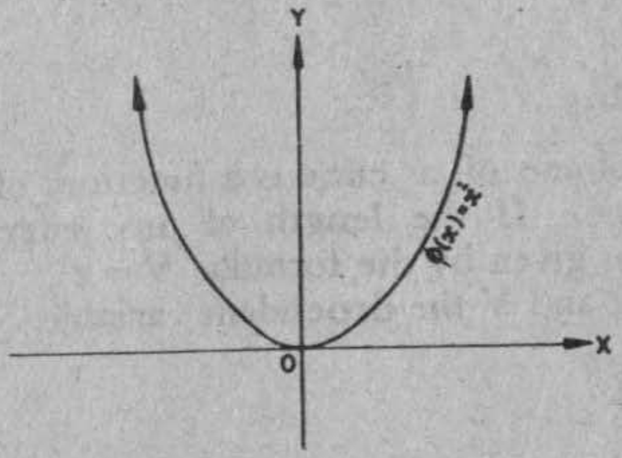


Fig. 2.5

Example 4. Draw the graph of the function

$$\varnothing(x)=x^2.$$

The graph of this function is shown in figure 2.6. The range of the function $\varnothing$ consists of only non-negative numbers and hence the graph never goes below x-axis. Such a function is said to be *non-negative.*



Incomplete Graph

Fig. 2.6

This figure is a parabola. The domain of this function is the set of all real numbers

Example 5. Discuss $x^2+y^2=r^2$, and draw its graph.

Solving this equation for y, we get

$$y=\sqrt{r^2-x^2} \text{ and } y=-\sqrt{r^2-x^2}$$

In such cases it was customary to say that y is a "*doube-valued function*" of x given by $y=\pm\sqrt{r^2-x^2}$. However the modern point of view does not admit the so called "*double-valuedness*" as a property of functions. The definition of a function requires that corresponding to each x in the domain there must be one and only one y in the range. Geometrically* it means that vertical lines which intersect the graph of the function should cut the graph at exactly one point. Thus in this example we say that the solutions of y define two functions, f and g, given by

$$f(x)=\sqrt{r^2-x^2} \text{ and } g(x)=-\sqrt{r^2-x^2}$$

---

*For details consult "Modern Algebra and Trigonometry."

Each of these functions has for its domain the interval extending from $-r$ to r. If $|x| > r$, there is no real y satisfying $x^2+y^2=r^2$, and we say that the functions f and g are *not defined* for such x.

To draw the graph we note that $f(x)$ is the positive square root of r^2-x^2 and hence the graph of f is the upper semi-circle shown in figure 2.7. The function $g(x)$ is the negative square root of r^2-x^2 and so the graph of g is the lower semi-circle shown in figure 2.8.

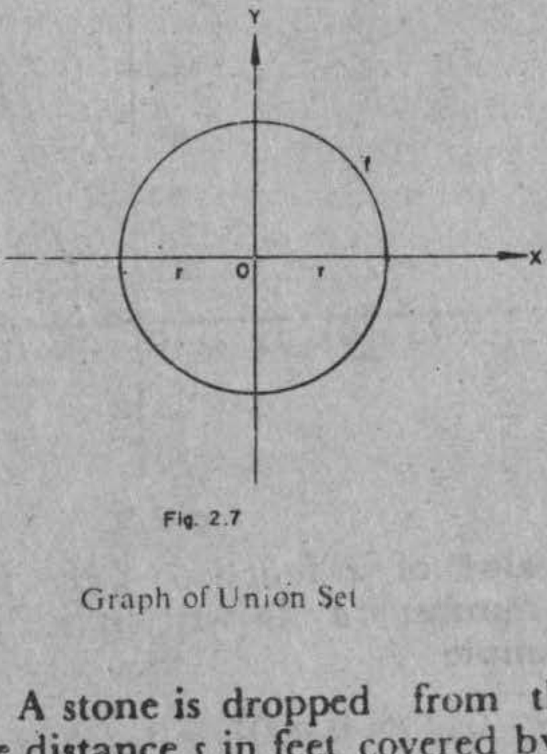


Fig. 2.7

Graph of Union Set

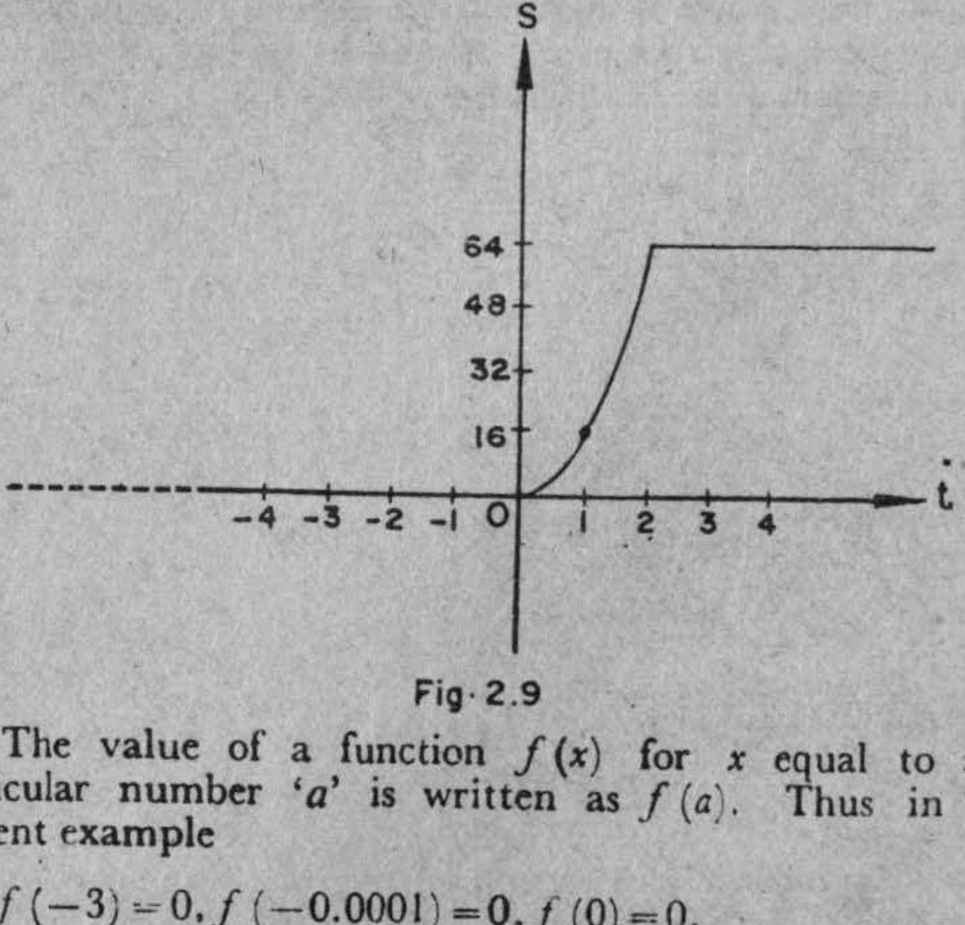
Example 6. A stone is dropped from the top of a vertical cliff. The distance s in feet covered by the stone is a function of the time t measured in seconds. Suppose we measure time and the distance from the instant the stone was released from the top of the cliff downward, *i.e.*, $s=o$ at $t=o$. Also suppose we know that s and t are related by the formula $s=16\,t.^2$ (We prove this formula later while studying Mechanics). Then the function

$$s=16\,t^2,\ t \geqq 0$$

gives the distance covered by the stone for all times after its release, provided it goes on falling and does not stop any where. But suppose the height of the cliff is only 64 ft. and so the stone strikes the base at $t=2$ sec. and lies there for ever after that. Then the "history" of the stone is given by the function

$$S=\begin{cases} 16\,t^2, & 0 \leqslant t < 2 \\ 64, & t \geqslant 2 \end{cases}$$

A graph of this function is given below.



Fig· 2.9

The value of a function $f(x)$ for x equal to any particular number 'a' is written as $f(a)$. Thus in the present example

$$\begin{aligned} &f(-3)=0, f(-0.0001)=0, f(0)=0, \\ &\quad f(1)=16, f(1.5)=36, \\ &f(1.7)=46.24, f(2.1)=64, f(3)=f(4)=f(4.3) \\ &\qquad =64, \text{ etc.} \end{aligned}$$

When we are required to study different functions of x at the some time it is better to use a different notation for each function to avoid confusion. If one function is donoted by $f(x)$, then the other may be written as $g(x)$, or $h(x)$, or $F(x)$, or $\psi(x)$, or $\varnothing(x)$, etc.

Example 7. We may specify a function without bothering about its physical significance. Consider the function.

$$g(x) = \begin{cases} 0, & x<0 \\ x, & 0 \leqslant x < 1 \\ 1-x, & 1 \leqslant x < 2 \\ 0, & x \geqslant 2 \end{cases}$$

Then $g(-5)=0$, $g(0)=0$, $g(.5)=0.5$, $g(1.4)=-0.6$

$g(1)=g(2)=g(3)=0$, etc.

The student is advised to draw a graph of this function, and then try to interpret it as a history of some natural phenomenon. This function may be interpreted as the history of an individual of a nation who was born in the world at a certain instant, gradually rose to fame or power or to any other measureable attribute, then had a similar gradual fall and finally disappeared for ever.

Example 8: Draw the graph of the function

$f(x) = |\operatorname{Sin} x|,\ 0 \leqslant x \leqslant 2\pi.$

The graph of the function in this example is shown below.

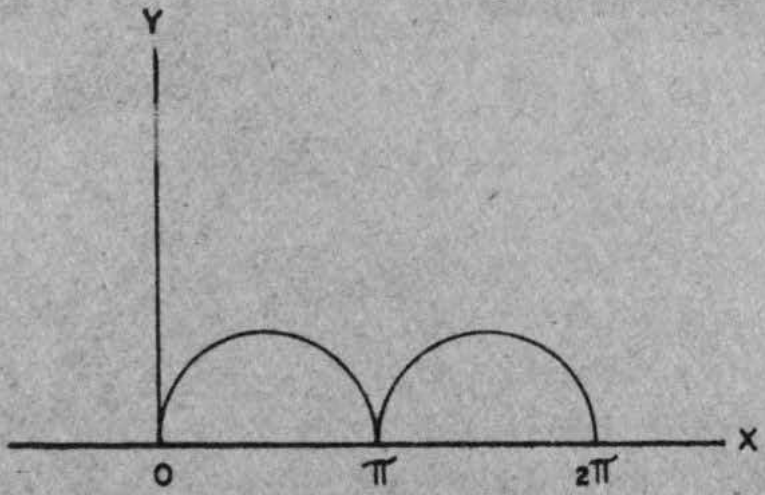


Fig. 2.10

Compare this graph with the graph of Sin x.

Exercise 2.1

1. Given that $f(x)=x^2+2x+3$ find $f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3)$. Plot graph of $f(x)$ for x between -3 and 3.

2. For each of the functions defined below ; state the domain imphed :

(*i*) $f(x)=(1-x^2)^{-1}$,

(*ii*) $g(x)=(1-x^2)^{\frac{1}{2}}$

(*iii*) $h(x)=(1-x^2)^{-\frac{1}{2}}$

(*iv*) $F(x)=(|x|-x)^{\frac{1}{2}}$,

(*v*) $G(x)=(|x|-x)^{-\frac{1}{2}}$

3. A function F is said to be even if $F(-x)=F(x)$, and odd if $F(-x)=-F(x)$, for every x in the domain of F. For each of the functions defined below ; state whether it is even, odd or neither ?

(*i*) $f(x)=x$,

(*ii*) $g(x)=|x|$,

(*iii*) $h(x)=x^2$,

(*iv*) $F(x)=\sin x^2$,

(*v*) $G(x)=x-x^2$,

(*vi*) $H(x)=x+x^3$,

(*vii*) $\phi(x)=|\sin x|$,

(*viii*) $\phi(x)=x+|x|$,

(*ix*) $K(x)=\dfrac{x}{|x|}$,

(*x*) $L.(x)=1-x^2$,

(*xi*) $M(x)=|1-x^2|$

4. If $f(x) = \text{Sin } x + \text{Cos } x$, show that

$$f(x+\pi) = -f(x).$$

5. If $\varnothing(x) = a^x$, show that

$$\varnothing(p) . \varnothing(q) = \varnothing(p+q).$$

6. If $\psi(x) = x^b$, show that $\psi(l)\,\psi(m) = \psi(lm)$.

7. If $g(x) = \log x$, prove that

$$(i)\ g(a\,b) = g(a) + g(b),$$

$$(ii)\ g\left(\frac{a}{b}\right) = g(a) - g(b),$$

$$(iii)\ g(a^b) = b\ g(a).$$

8. If $f(y) = \left(\frac{y+1}{y-1}\right)$, show that

$$f[f(y)] = y.$$

9. If $f(\theta) = \tan\theta$, show that

$$f(x-y) = \frac{f(x) - f(y)}{1 + f(x)\,f(y)}$$

10. Show that $\cos x$ is an even function of x.

11. Show that $\frac{e^x - 1}{e^x + 1}$ is an odd function of x.

12. If $f(x)$ is any function, show that $F(x) = f(x) + f(-x)$ is always an even function, and $G(x) = f(x) - f(-x)$ is always an odd function.

13. Draw the graph for each of the functions in problem 3. What criterion do you find for recognizing an even or an odd function from the shape of the graph?

14. A square is inscribed in a circle of radius r. Express the side of the square x as a function of r.

15. A rectangle is inscribed in a circle of radius 5 dm. If the lengths of the sides of the rectangle are $2x$ and $2y$, express (*i*) y as a function of x, (*ii*) the area of the rectangle as a function of x.

16. A man 6 dm. tall is walking away from a lamp post the height of the lamp post being 9 dm. Let y be the length of the man's shadow and z the distance of the tip of the shadow from the foot of the post. When the man is at a distance x from the post, express y and z as a function of x.

17. An open box is to be made by cutting out squares from the corners of a rectangular piece of card board and then turning up the sides. If the piece of card board is 18 cm. by 30 cm. express the volume of the box as a function of its height h.

18. A man in a boat is at 4 km. from the nearest point P on the straight shore line. Taking P as the origin and the shore line as x-axis, the present position of the man is $(0, 4)$. He rows in a straight line at 4 km/hr. to land at the point $Q(x, 0)$ and then runs to the point $R(6, 0)$ along the shore at 6 km/hr. Find the total time to reach R as a function of x.

19. Give examples of functions satisfying the following (functional) equations for all real numbers x and y:

(*i*) $f(2x)=2f(x)$,

(*ii*) $g(x)+g(y)=g(x+y)$,

(*iii*) $h(x)\,h(y)=h(x+y)$,

(*iv*) $F(xy)=F(x)+F(y)$,

(*v*) $G(2x)=\dfrac{2G(x)}{1-G^2(x)}$,

(*vi*) $H(2x)=2H(x)\sqrt{1-H^2(x)}$.

ANSWERS

1. $6, 3, 2, 3, 6, 11, 18.$

2. (*i*) all except $x = \pm 1$. (*ii*) $|x| \leqslant 1$.
(*iii*) $|x| < 1$. (*iv*) $\forall x$.
(*v*) $\forall x < 0$.

3. (*i*) odd. (*ii*) even.
(*iii*) even. (*iv*) even.
(*v*) neither. (*vi*) odd.
(*vii*) even. (*viii*) neither.
(*ix*) odd. (*x*) even.
(*xi*) even.

14. $x = \sqrt{2}\, r$. 15. $y = \sqrt{25 - x^2}$.

16. $y = 2x, \quad z = 3x$. $A = 4x\sqrt{25 - x^2}$.

17. $V = h(30 - 2h)(18 - 2h)$.

18. $T = \dfrac{\sqrt{16 + x^2}}{4} + \dfrac{|6 - x|}{6}$

19. (*i*) $f(x) = kx$
(*ii*) $g(x) = kx$.
(*iii*) $h(x) = a^x$.
(*iv*) $F(x) = \log x$.
(*v*) $G(x) = \tan x$.
(*vi*) $H(x) = \sin x$.

2.4. *Meaning of the phrase "x tends to a".*

Suppose that a variable x takes in succession the set of values:

$$a+\frac{1}{10},\ a+\frac{1}{10^2},\ a+\frac{1}{10^3},\ \ldots\ldots, a+\frac{1}{10^n},\ \ldots\ldots\infty\ (n\in N)$$

Then we note the following behaviour of x in relation to the number 'a'. The successive differences between the values of x and a are

$$\frac{1}{10},\ \frac{1}{10^2},\ \frac{1}{10^3},\ \ldots\ldots\ldots\ldots\ldots,\frac{1}{10^n},\ \ldots\ldots\ldots\ldots,$$

which go on becoming smaller and smaller as n increases. In fact this difference can be made as small as we like by taking 'n' sufficiently large. Thus a stage in the variation of x can be reached after which the difference between any value of x and a is less than any given positive number, however small. This behaviour of x is symbolically expressed as "$x\to a$" and read as "x tends to a" or "x approaches a".

If a variable quantity x takes in succession the set of values:

Example 1. $1+\frac{1}{10}, 1+\frac{1}{10^2}, 1+\frac{1}{10^3}, \ldots\ldots\ldots\infty$;
then $x\to 1$.

Example 2. $3, 3\frac{1}{2}, 3\frac{2}{3}, 3\frac{3}{4}, 3\frac{4}{5}, 3\frac{5}{6}, \ldots\ldots\infty$; then $x\to 4$.

Example 3. $(2-\frac{1}{3}), \left(2-\frac{1}{3^2}\right), \left(2-\frac{1}{3^3}\right), \ldots\ldots$;
then $x\to 2$.

Example 4. $\frac{1}{5}, \frac{1}{5^2}, \frac{1}{5^4}, \frac{1}{5^8}, \ldots\ldots\infty$; then $x \to 0$.

From the above examples, it is clear that a variable x is said to tend to a fixed number 'a', if whenever x takes in succession a sequence of values $a_1, a_2, a_3, \ldots\ldots\ldots\ldots\infty$, then the stage in the variation of x can be reached after which the difference between each value of x and a is less than any given positive number, however small. If x takes in succession a set of values $c_1, c_2, c_3, \ldots\ldots\ldots, \infty$ and if a stage in the variation of x can be reached after which the values of x are greater than any given positive number, however large, then the variable is said to "tend to ∞." In symbols we write $x \to \infty$.

In the following examples $x \to \infty$, if it takes in succession the values :

(*i*) $1, 2, 3, 4, 5, \ldots\ldots\ \infty$

(*ii*) $2+\frac{1}{2}, 4+\frac{1}{4}, 6+\frac{1}{6}\ 8+\frac{1}{8}, \ldots\ldots\ldots\infty$

(*iii*) $3, 3^2, 3^3, 3^4, 3^5, \ldots\ldots\ldots\infty$.

The phrase "$f(x) \to l$ as $x \to a$".

Let x takes in succession a set of values $a_1, a_2, a_3 \ldots$, so that "x tends to a" through these values. The corresponding values of the function $f(x)$ are $f(a_1)$, $f(a_2)$, $f(a_3)$, ... Suppose that these values of $f(x)$ tend to a fixed quantity l (say), in the same sense as x tends to a. Then l is called the limit of $f(x)$ as x tends to a. It is symbolically written as :

$$\lim_{x \to a} f(x) = l$$

2.5. *Theorems on Limits.*—We state below some of the important theorems on limits. The proofs are omitted, since they are beyond the scope of this book.

(*i*) $\lim\limits_{x \to a} [f(x) \pm \varnothing(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to o} \varnothing(x)$.

(*ii*) $\lim\limits_{x \to a} cf(x) = c \lim\limits_{x \to a} f(x)$; where c is a constant.

(*iii*) $\lim\limits_{x \to a} [f(x) . \phi(x)] = \lim\limits_{x \to a} f(x) . \lim\limits_{x \to a} \phi(x)$

(*iv*) $\lim\limits_{x \to a} \left[\dfrac{f(x)}{\phi(x)}\right] = \lim\limits_{x \to a} f(x) / \lim\limits_{x \to a} \phi(x)$; provided

$$\lim_{x \to a} \phi(x) \neq 0.$$

2.6. *The Limit and the Value of a Function.*

Consider the following examples :

(*a*) If $f(x) = x$, then $\lim\limits_{x \to 1} f(x) = 1$

Also the value of $f(x)$ at $x = 1$ is given by $f(1) = 1$. Thus in this case the limit of the function $f(x)$ is equal to its value.

(*b*) If $f(x) = \dfrac{x^2-4}{x-2}$, then $\lim\limits_{x \to 1} \dfrac{x^2-4}{x-2}$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \to 2} (x+2) = (2+2) = 4.$$

($\because$ $x \neq 2$, so we can divide by the factor $x-2$, which is not equal to zero) on the other hand if we put $x = 2$ in $f(x)$, we have $f(2) = \dfrac{2^2-4}{2-2} = \dfrac{4-4}{2-2} = \dfrac{0}{0}$. Hence in this case the value of the function does not exist. Also it can be easily proved that for any other value of x, both the value and the limit of the function exist and are equal.

Again there are functions which, for a certain value of x have a value but their limits do not exist. Also there are functions having both a limit and a value, but the value is not equal to the limit.

Example 1. Find the limit of $f(x) = x^2 + 2$ when $x \to 1$.

Solutioin—For finding limit of a function $f(x)$ when $x \to a$ we will use the following working rule :

In the function $f(x)$, put $x = a+h$. Then the limit of $f(x)$ when $x \to a$ is the limit of $f(a+h)$ when $h \to 0$.

In this example $f(x) = x^2+2$, $a = 1$

$$\therefore \qquad f(a+h) = f(1+h)$$

$$= (1+h)^2+2$$

$$= 3+2h+h^2$$

$$\therefore \qquad \lim_{x \to 1} f(x) = \lim_{h \to 0} 3+2h+h^2 = 3.$$

Example 2. Find $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2}$.

Solution, Here we have $f(x) = \dfrac{x^2-4}{x-2}$

$$\therefore \qquad \lim_{x \to 2} f(x) = \lim_{h \to 0} \frac{(2+h)^2-4}{2+h-2}$$

$$= \lim_{h \to 0} \frac{4h+h^2}{h}$$

$$= \lim_{h \to 0} (4+h) = 4.$$

It may be pointed out that in example 1,

$$\lim_{x \to 1} f(x) = 3 = f(1).$$

$$= \text{value of } f(x) \text{ at } x = 1.$$

In example 2,

$$\lim_{x \to 2} f(x) = 4.$$

$$f(2)=\frac{4-4}{2-2}=\frac{0}{0} \text{ which is not defined.}$$

Here $\lim\limits_{x\to 2} f(x)$ exists, but the value of the function at $x=2$ does not exist.

Example 3. For $f(x)=\dfrac{\sqrt{4+x}-2}{x}$ find $\lim\limits_{x\to 0} f(x)$.

Multiplying both the numerator and the denominator by

$$\sqrt{4+x}+2,$$

$$f(x)=\frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{x(\sqrt{4+x}+2)}$$

$$=\frac{4+x-4}{x(\sqrt{4+x}+2)}=\frac{1}{\sqrt{4+x}+2},\ x\neq 0.$$

$$\text{When } x\to 0, f(x)\to \frac{1}{\sqrt{4+0}+2}=\tfrac{1}{4}.$$

Hence $\lim\limits_{x\to 0} f(x)=\frac{1}{4}$.

Example 4. Evaluate $\lim\limits_{x\to\infty} \dfrac{2x^3+4x^2+5}{3x^3+x^2-6}$.

Solution. Put $x=\dfrac{1}{y}$. Then as $x\to\infty$, $y\to 0$.

$$\therefore \lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{2x^3+4x^2+5}{3x^3+x^2-6}$$

$$=\lim_{y\to 0}\frac{2\left(\frac{1}{y}\right)^3+4\left(\frac{1}{y}\right)^2+5}{3\left(\frac{1}{y}\right)^3+\left(\frac{1}{y}\right)^2-6}$$

$$=\lim_{y\to 0}\frac{2+4y+5y^3}{3+y-6y^3}=\frac{2}{3}$$

Exercise 2.2

1. In each of the following problems, a_n of a sequence is given. Find their limits if any.

(i) $a_n=\dfrac{n-1}{n+1}$,

(ii) $a_n=\dfrac{n}{2n-1}$,

(iii) $a_n=1/n^2$,

(iv) $a_n=n/(1+n^2)$,

(v) $a_n=2+3/n$,

(vi) $a_u=n^2/(2n+1)-n^2/(2n-1)$,

(vii) $a_n=(1^2+2^2+3^2 \ldots\ldots\ldots +n^2)/n^3$,

(viii) $a_n=(1/n)^n$,

(ix) $a_n=\sqrt{n+2}-\sqrt{n+1}$,

(x) $a_n=\dfrac{n^2}{n+1}$.

2. Evaluate the following limits :—

(i) $\lim\limits_{h\to 0}\dfrac{(1+h)^2-1}{h}$,

(ii) $\lim\limits_{x\to 1}\dfrac{\sqrt{x-1}}{x-1}$,

(iii) $\lim\limits_{h\to 0}\dfrac{(2+h)^2-2^2}{h}$,

(iv) $\lim\limits_{h\to 0}\dfrac{\sqrt{9+h-3}}{h}$,

(v) $\lim\limits_{x\to \infty}\dfrac{5x^2+3x+4}{3x^2+7}$,

(vi) $\lim\limits_{x\to 0}\dfrac{5x^2+3x+4}{3x^2+7}$, (*vii*) $\lim\limits_{x\to 0} \dfrac{x+1}{x^2+x+1}$,

(*viii*) $\lim\limits_{x\to \infty} \dfrac{x+1}{x^2+x+1}$,

(*ix*) $\lim\limits_{x\to \infty} \dfrac{x(x+1)}{(x+2)(x+4)}$,

(*x*) $\lim\limits_{n\to \infty} \dfrac{n(n+1)(2n+1)}{n^3}$,

(*xi*) $\lim\limits_{n\to \infty} \dfrac{[n(n+1)]^2}{n^4}$.

ANSWERS

1. (*i*) 1 (*ii*) $\frac{1}{2}$
(*iii*) 0 (*iv*) 0
(*v*) 2 (*vi*) $-\frac{1}{2}$
(*vii*) $\frac{1}{3}$ (*viii*) limit does not exist.
(*ix*) 0 (*x*) limit does not exist.

2. (*i*) 2 (*ii*) $\frac{1}{2}$
(*iii*) 4 (*iv*) $\frac{1}{6}$
(*v*) $\frac{5}{3}$ (*vi*) $\frac{4}{7}$
(*vii*) 1 (*viii*) 0
(*ix*) 1 (*x*) 2
(*xi*) 1

A sequence which possesses a limit is called *convergent* and which does not possess a limit is called *divergent*. Sequences given in question 1 (*i*)—1 (*vii*) and 1 (*ix*) are examples of convergent sequences, whereas questions 1 (*viii*) and 1 (*x*) are examples of divergent sequences.

2.7. *Some Important Limits*

(*i*) $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}$, where a is any number and n is a positive integer. By ordinary long division we easily find that $\dfrac{x^n-a^n}{x-a}=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots\ldots+a^{n-2}x+a^{n-1}$.

As $x\to a$, each term on the right hand approaches a^{n-1}. But the total number of terms on the right hand side is n.

Thus

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}.$$

This result* may also be obtained as follows by writing $x=a+h$, then using the Binomial Theorem and finally letting $h\to 0$:

$$\frac{x^n-a^n}{x-a}=\frac{(a+h)^n-a^n}{h}$$

$$=\frac{1}{h}\Big[a^n+na^{n-1}h+\frac{n(n-1)}{2\,!}a^{n-2}h^2+\ldots\ldots\ldots\ldots\ldots$$
$$+nah^{n-1}+h^n-a^n\Big]$$

$$=na^{n-1}+\frac{n(n-1)}{2\,!}a^{n-2}h+\text{terms containing } h^2 \text{ and}$$

higher powers.

$$\text{Hence }\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{h\to 0}\Big[na^{n-1}+\frac{n(n-1)}{2\,!}a^{n-2}h+\ldots$$
$$\ldots\ldots\ \ldots\ldots+h^{n-1}\Big]=na^{n-1}.$$

(*ii*) $\lim\limits_{x\to 0}\dfrac{\operatorname{Sin} x}{x}\Big[x \text{ is measured in radians}\Big]$.

---

* The result is also true if n is a rational number.

We first like to have an idea of the value of the ratio $\frac{\text{Sin } x}{x}$ for small x, when x is measured in radians. To do this we use the table of values of sines available to us. Remembering that 1 radian = 57°17′45″ approximately, we have 0.5 radian = 28°38′53″, 0·2 radian = 11°27′31″, 0.1 radian = 5°43′46″, 0.05 radian = 2°51′53″; and 0.01 radian = 0°34′23″ approximately. Accordingly the following table of approximate values is obtained.

x(radian)	0.5	0.2	0.1	0.05	0.01
sin x	0.4795	0.1987	0.0998	0.04998	0.009999
$f(x) = (\sin x)/x$	0.959	0.995	0.998	0.9996	0.9999

It looks as if the limit required is 1. We now prove the result :

$$\lim_{x \to 0} (\text{Sin } x)/x = 1.$$

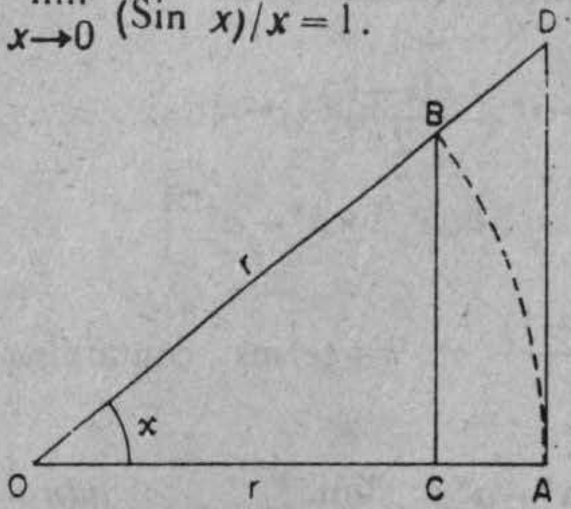


Fig.2.11

Let AB be an arc of a circle of radius r and centre O., Also suppose that $\angle AOB = x$ radians, where x is small.

Draw $BC \perp OA$ meeting OA in C and $AD \perp OA$ meeting OB produced in D. Then $CB = r \sin x$, $AD = r \tan x$, are $AB = r x$.

Area of $\triangle$ OAB $= \frac{1}{2}$ (OA) (BC) $= \frac{1}{2} r^2 \sin x$.

Area of sector O A B $= \frac{1}{2} r^2 x$.

Area of Δ OAD $= \frac{1}{2}$ (OA) (AD) $= \frac{1}{2} r^2 \tan x$.

Now it is evident from the figure that :

Area of Δ OAB $<$ Area of sector O A B $<$ Area of Δ O A D.

Thus $\frac{1}{2} r^2 \sin x < \frac{1}{2} r^2 x < \frac{1}{2} r^2 \tan x$, so that

$$\sin x < x < (\sin x)/(\cos x).$$

Dividing throughout by $\sin x$, which is positive and does not change the sense of the inequalities, we get :

$1 < x/(\sin x) < 1/(\cos x)$

i. e. $1 > (\sin x)/x > \cos x$.

Now, as $x \to 0$, $\cos x \to 1$. This shows that $\frac{\sin x}{x}$ lies between 1 and a quantity which tends to 1, and so $(\sin x)/x$ must itself tend to 1.

Hence $\lim\limits_{x \to 0} \sin x/x = 1$.

Example 1.—$\lim\limits_{x \to 0} \left(\frac{x}{x+1} + \frac{x+1}{x-1} \right)$

$$= \lim_{x \to 0} \frac{x}{x+1} + \lim_{x \to 0} \frac{x+1}{x-1} = 0 - 1 = -1$$

Example 2.—$\lim\limits_{x \to 3} \frac{x^2-9}{x-3} = \lim\limits_{x \to 3} \frac{(x+3)(x-3)}{x-3}$,

dividing the numerator and denominator by $(x-3)$ which is not zero we obtain

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} (x+3) = 3+3 = 6.$$

Example 3.—Calculate $\lim\limits_{x \to 0} \frac{\sin (px)}{x}$.

$\frac{\sin (px)}{x} = p . \frac{\sin (px)}{px} = p \frac{\sin z}{z}$ where $z = px$ and $z \to 0$ as $x \to 0$.

$$\therefore \lim_{x \to 0} \frac{\sin (px)}{x} = \lim_{z \to 0} p \frac{\sin z}{z}$$

$$= p \lim_{z \to 0} \frac{\sin z}{z} = p . 1 = p.$$

Example 4 — $\lim_{x \to 0} \frac{\tan x}{x}$

$$\frac{\tan x}{x} = \frac{\sin x}{x} \frac{1}{\cos x}$$

$$\therefore \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x} . \lim_{x \to 0} \frac{1}{\cos x} = 1.1 = 1.$$

Exercise 2.3

Evaluate :

1 $\lim_{x \to 2} \frac{x^4 - 16}{x - 2}$

2. $\lim_{x \to 1} \frac{x^5 - 1}{x^1 - 1}$

3. $\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n}$

4. $\lim_{x \to 1} \frac{x^{\frac{2}{3}} - 1}{x^{\frac{3}{5}} - 1}$

5. $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 + 2x - 8}$

6. $\lim_{x \to} 3 \sqrt{\frac{x^2 - 7x + 12}{x^2 - 10x - 21}}$

7. $\lim_{x \to 0} \frac{\sin px}{x}$

8, $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta}$

9. $\lim\limits_{\theta \to 0} \frac{1 - \cos \theta}{\theta^2}$

10. $\lim\limits_{\theta \to 0} \frac{1 - \cos \theta}{\sin \theta}$

11. $\lim\limits_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$

12. $\lim\limits_{\theta \to 0} \frac{\tan p\theta}{\theta}$

13. $\lim\limits_{\to \frac{\pi}{2}} (\sec x - \tan x)$

14. $\lim\limits_{\theta \to 0} \frac{\sin \theta^\circ}{\theta^\circ}$

15. $\lim\limits_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

16. $\lim\limits_{x \to 0} \frac{\sqrt{x-p} - \sqrt{p}}{x}$

17. $\lim\limits_{o \to 0} \frac{\theta}{1 - \sqrt{1-\theta}}$

18. $\lim\limits_{\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

19. $\lim\limits_{z \to 0} \frac{1 - \cos p\phi}{1 - \cos q\phi}$

20. $\lim\limits_{\downarrow \to 0} \frac{\tan \downarrow - \sin \downarrow}{\sin^3 \downarrow}$

ANSWERS

1. 32 2. $\frac{5}{3}$

3. $\frac{m}{n} a^{m-n}$ 4. $\frac{10}{9}$

5. $-1/6$ 6. $1/2$

7. p
8. 0
9. $\frac{1}{2}$
10. 0
11. $\frac{1}{2\sqrt{a}}$
12. p
13. 0
14. $\frac{\pi}{180}$
15. 1
16. $\frac{1}{2\sqrt{p}}$
17. 2
18. $\frac{1}{2}$
19. $\frac{p^2}{q^2}$
20. $\frac{1}{2}$

2.8. *Continuity* : The statement that a certain function $f(x)$ of x is defined in some interval $[a, b]$ means that corresponding to every value of x in this interval there is a value of the function. It is to be noted that the value of the function for a specified value of x may be quite independent of the value of the function for some other value of x. As a matter of fact there may be no relationship in the various values of a function.

Thus, if x_1, x_2 are any two values of the independent variable x and $f(x_1), f(x_2)$ are the corresponding values of the function $f(x)$, then $|f(x_2)-f(x_1)|$ may be large even though $|x_2-x_1|$ may be small. This is because the value $f(x_2)$ of the function for $x=x_2$ is quite independent of the value $f(x_1)$ of the function for $x=x_1$. Our main aim is to study the change $|f(x_2)-f(x_1)|$ relative to the change $|x_2-x_1|$ by introducing the notion of *continuity* and *discontinuity* of a function.

Intuitively, when we say that a function is *continuous*, we ordinarily mean that its graph has no *sudden jumps* or breaks. The function $f(x)$ will be *continuons at the point* $x=c$ if as x varies from 'c' to either side of it, the change in the value of the function is not sudden. Alternatively, a function $f(x)$ is said to be *continuous at* $x=c$ if and only if

$$\lim_{x \to c} f(x) = f(c).$$

In other words, a function $f(x)$ is continuous at a point $x=c$ if the limit of the function at $x=c$ is equal to the value of the function at $x=c$. This statement is not flawless but it is not possible here to give a precise definition.

A function which is not continuous at a point is said to be discontinuous at that point. The students should note that the continuity and discontinuity of a function is always discussed at a point.

Example 1. Show that the function $4x^3+1$ is continuous at $x=2$.

$$\text{Let } f(x)=4x^3+1$$
$$f(2)=4.2^3+1=33$$

Again put $x=2+h$; so that $h \to 0$ when $x \to 2$.

$$\text{Now } \lim_{x \to 2} f(x) = \lim_{x \to 2} (4x^3+1) = \lim_{h \to 0} 4\,(2+h)^3+1=33$$

Thus $f(2) = \lim\limits_{x \to 2} f(x)$. Hence the function $4x^3+1$ is continuous at $x=2$.

It is to be noted that this function is continuous for every finite real value of x.

Example 2. Prove that the function $\frac{x^2-9}{x-3}$ is discontinuous at $x=3$.

Let $f(x) = \frac{x^2-9}{x-3}$. Put $x=3+h$; so that $h \to 0$, as $x \to 3$.

$$\text{Now } \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{h \to 0} \frac{(3+h)^2-9}{3+h-3}$$

$$= \lim_{h \to 0} \frac{9+6h+h^2-9}{h} = \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} (6+h) = 6.$$

Again $f(3)=\frac{9-9}{3-3}=\frac{0}{0}$. The function is indeterminate at $x=3$ and so its value does not exist at $x=3$.

Thus $f((3)\neq \lim\limits_{x\to 3} f(x)$, Hence the function $\frac{x^2-9}{x-3}$ is discontinuous at $x=3$.

Write down all the values of x for which this function is continuous.

Example 3. Prove that the function Sin x is continuous in the interval $[0, \pi]$.

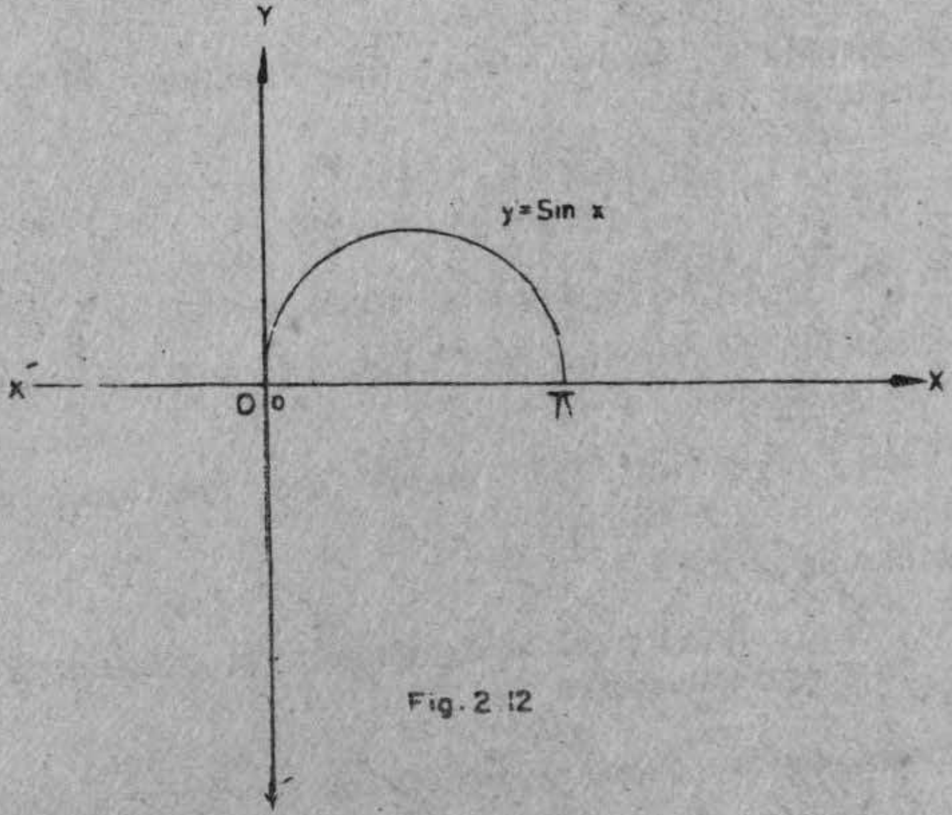


Fig. 2 12

A look at the graph of Sin x is enough to show that there is no break or gap in the graph in the interval $[0, \pi]$ and so Sin x is continuous in this interval. Alternatively it is easy to verify that the value of this function at every point of this interval is equal to the corresponding limit of the function and hence the function Sin x is continuous in the interval [0,]

Example 4. Show that the function tan x is discontinuous at the point $x=\pi/2$.

The students are fully aware that the value of tan x can not be found at $x=\pi/2$, whereas lim tan $x=\infty$. That is, the limit does not exist,

$x \to \pi/2$

Hence tan x is discontinuous at $x=\pi/2$.

Write down three points in the interval $[0, \pi/2]$ at which tan x is continuous. Verify this by actual calculation.

Exercise 2.4

Show that the following functions are continuous at every point in the domain of real numbers :

1. x
2. $(x+2)^2$
3. $2x^4-7$
4. Cos x
5. Sin $(\theta+3)$
6. Sin θ Cos θ.

Show that the following functions are discontinuous at the points mentioned against each.

7. $\dfrac{x^2-16}{x-4}$ at $x=4$
8. $\dfrac{x^n-a^n}{x-a}$ at $x=a$
9. Cot x at $x=0$
10. Sec x at $x=\dfrac{\pi}{2}$

Find out the points at which the following functions are discontinuous in the intervals mentioned against each.

11. $\dfrac{x^3-1}{x^4-1}$; $[-5, 5]$
12. $\sqrt{1+x}-1$; $[-1, 10]$

13. $\text{Cosec}\, 0 \,;\, [-2\pi,\ \pi]$

14. $\text{Cot}\, x \,;\, [-2\pi,\ 2\pi]$

15. $\dfrac{1 \quad \text{Cos}\, x}{1-\text{Cos}\, 2x}; \left[\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$

16. $\dfrac{\text{Cosec}\, x - \text{Cot}\, x}{x}; \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$

17. $\dfrac{1}{x}\left[\dfrac{1}{\sqrt{x+a}} - \dfrac{1}{\sqrt{a}} \right]; [-a, a],$

18. $\dfrac{\text{Sin}\, x}{\text{[illegible]}}; [0, \pi].$

ANSWERS

11. 1 — 12. 0

13. $-2\pi, 0, \pi.$ — 14 $-2\pi, -\pi, 0, \cdot, 2\pi$

15. 0 — 16. 0

17. 0 — 18. 0

CHAPTER III

DIFFERENTIATION AND INTEGRATION OF ALGEBRAIC FUNCTIONS

3.1 *Increments.* Since we are going to study the changes in the quantities x and y for a function $y = f(x)$, x and y are called variables ; x the independent variable and y the dependent variable. Consider the function,

$$f(x) = x^2$$

We note that $f(1) = 1$, $f(2) = 4$, $f(3) = 9$ etc. This shows that as x changes, $f(x)$ also changes. when x receives an increment of 1 at $x = 2$, corresponding increment in $f(x)$ is

$$f(3) - f(2) = 9 - 4 = 5.$$

Similarly when x changes from a to $a + h$, $f(x)$ changes from $f(a)$ to $f(a + h)$ and so corresponding to the increment h in x at $x = a$, the increment in $f(x)$ is

$$f(a+h) - f(a).$$

We denote an increment in x by Δx and the corresponding increment in $f(x)$ by Δf.

The table below gives the value of Δf corresponding to the values of x and Δx.

x	$f(x)$	Δx	Δf
0	0	1	1
1	1		
1.5	2.25	.5	1.25
2	4	.5	1.75
2.4	5.76	.4	1.76
2.5	6.25	.1	.49

For all values of x and Δx, we can compute Δf in the following manner :—

$$\begin{aligned} f(x) = x^2,\ \Delta f &= f(x + \Delta x) - f(x) \\ &= (x + \Delta x)^2 - x^2 \\ &= 2x\,\Delta x + (\Delta x)^2 \ldots\ldots\ldots\ldots\ (A) \end{aligned}$$

From the above result (A) compute Δf for the following cases :—

$$\begin{aligned} x &= 3,\quad \Delta x = 0.1 \\ x &= 3,\quad \Delta x = 0.01 \\ x &= 3,\quad \Delta x = 0.001 \\ x &= 3,\quad \Delta x = 0.0001 \\ x &= 3,\quad \Delta x = 0.0000000001 \end{aligned}$$

We abserve that when Δx, is very small, then $(\Delta x)^2$ is almost negligible. Thus for small Δx, the equation (A) becomes $\Delta f = 2x\,\Delta x$. For small increments in x and $f(x)$, we will use the symbols δx and δf.

Example 1. The volume of a sphere of radius x is given by $V = \frac{4}{3}\pi x^3$ A spherical balloon of radius 2 decimeters expands due to decrease in external pressure. If the radius increase by 3 centimeters find the increase in the volume.

Solution : $V = \frac{4}{3}\pi x^3,\ x = 20,\ \Delta x = 3.$

$$\begin{aligned} V + \Delta V &= \tfrac{4}{3}\pi (x + \Delta x)^3. \\ \Delta V &= (\tfrac{4}{3})\pi\,[(x + \Delta x)^3 - x^3] \\ &= (\tfrac{4}{3})\pi\,[3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3]. \end{aligned}$$

Taking $x = 20,\ \Delta x = 3,$

$$\begin{aligned} \Delta V &= 4\pi\,[(20)^2(3) + 20(3)^2 + \tfrac{1}{3}(3)^3] = 5556\pi \\ &= 12961.7 \text{ cu. cm.} = 17461.7 \text{ c. cm.} \end{aligned}$$

Example 2. Solve example 1 if there is a negative increment, *i.e.*, decrease of 3 centimeters in the radius.

Solution : As before,

$$\Delta V = 4\pi\,[x^2(\Delta x) + x(\Delta x)^2 + \tfrac{1}{2}(\Delta x)^3]$$

Here $x = 20,\ \Delta x = -3.$

$\therefore$ **Increment in volume**

$$= \Delta V = 4\pi \left[(20)^2 (-3) + 20 (-3)^2 + \frac{1}{2} (-3)^3\right]$$

$$= -4116\,\pi = -12936 \text{ cu. cm.} = -13 \text{ cu. dm.}$$

The negative increment shows that there is a decrease of 13 c.dm. in volume.

Example 3. Solve example 1 for a change of .01 cm. in radius.

Solution : Since the change in x is small, we denote it by δx.

$\therefore \quad \delta V = \frac{4}{3} \pi \left[(3x^2 (\delta x) + 3x (\delta x)^2 + (\delta x)^3\right]$. Neglecting $(\delta x)^2$ and $(\delta x)^3$, we get

$$\delta V = 4\pi x^2 (\delta x) = 4\pi (400)(.01) = 16\,\pi = 50.3 \text{ cu. cm.}$$

$$= 0.05 \text{ cu. dm.}$$

Exercise 3.1

1. If the radius of a sphere increases by $0 \cdot 1\%$, show that the volume increases by about $0 \cdot 3\%$.
2. For the function $f(x) = 1/x$, $x > 0$, find δf when x increases from 8 to 8.1.
3. For the function in problem 2 find a general expression δf for a small increment δx in x.
4. For the function $S = kt^2$, k a constant, $0 < t < 3$, find δs at $t = 2$, for $\delta t = 0.1$, 0.01, 0.001, 0.0001 etc. Construct a table giving the values of δt, δs and $\delta s/\delta t$.
5. For the function of problem 4, find an approximate expression giving the value of the ratio $\delta s/\delta t$.

6. Consider the function $y = \sin x$, where the angle x is measured in radians. At $x = 1{\cdot}0472$, find $\triangle y$ for $\triangle x = 0{\cdot}007$. Also compute $\delta y/\delta x$.

[Hint-Use tables, noting that 1·0472 radians $= 60°$ and ·007 radian $= 0°24'$]

7. Repeat problem 6 with the function $y = \cos x$, $x = .6984$ (*i.e.* 40°), $\delta x = 0.007$.

ANSWERS

2. $-0{\cdot}00156$.

5. $2\,kt$.

7. $\triangle y = -0{\cdot}0045, \dfrac{\triangle y}{\triangle x} \doteq -0{\cdot}6426$.

3.2. *Rate of Change*—In order to explain the concept of the rate of change we take example of the speed of an object moving in a straight line. It is commonly understood that the speed of an object is given by the expression.

$$\frac{s_1 - s}{t_1 - t} = \frac{\Delta s}{\Delta t},$$

where s, s_1 denote the distances covered by the object from a fixed point in the line in times, t, t_1 respectively. But precisely speaking the above expression gives only the average speed of the object from time t to t_1 in case the object does not traverse equal distances in equal intervals of time. In order to obtain the speed at a particular instant we have to investigate the behaviour of the ratio $\dfrac{\Delta s}{\Delta t}$ as Δt becomes smaller and smaller. If the ratio $\dfrac{\Delta s}{\Delta t}$ tends to a definite value as Δt approaches zero, we define this definite value to be the instantaneous speed at the instant t.

Generally, in case of a function $y=f(x)$, if the ratio $\frac{\delta y}{\delta x}$ tends to a definite value as δx approaches zero, we define this definite value to be the rate of change of the function y w.r.t.* x at the point x.

Example—For $y=16x^2$, find an expression for the rate of change of y w.r.t. x. Calculate its value at $x=1,2, 2.5$.

Solution. $y=kx^2$, where $k=16$.

$$y+\delta y=k(x+\delta x)^2=k[x^2+2x\delta x+(\delta x)^2]$$

$$\Rightarrow \delta y=k(x+\delta x)^2-kx^2=2kx.\ \delta x+k(\delta x)^2$$

$$\Rightarrow \frac{\delta y}{\delta x}=2kx+k\delta x.$$

We could now see that when δx becomes smaller and smaller and approaches zero, then $\frac{\delta y}{\delta x}$ approaches the value $2kx$ *i.e.* $32\,x$, which is the rate o change at any point x. The particular values at $x=1,2,\ 2{\cdot}5$ are 32, 64, 80 respectively.

Before speaking more precisely about the definite value to which the ratio $\frac{\delta y}{\delta x}$ tends, the students should note that the concept of limit of a function plays a very important role in this. The following procedure explains this in detail.

3.3. *The Derivative*—We now turn to the idea of "rate of change" developed in § 3.2. We have seen there that the following four steps give the rate of change w.r.t. x for the function $y=f(x)$.

Step 1. $y+\delta y=f(x+\delta x)$ (giving increment δx to x).

Step 2. $\delta y=f(x+\delta x)-f(x)$ (by subtraction).

---

* w.r.t. stands for 'with respect to.'

Step 3. $\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$ (dividing by δx).

Step 4. $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$

(proceeding to limit).

The result of step 4, provided the limit exists, is called the *Derivative* or the *Differential Coefficient* of y w. r. t. x. The derivative of y w. r. t. x is denoted by symbols Dy, $D_x y$, $\frac{d}{dx}(y)$, or $\frac{dy}{dx}$. The last symbol $\frac{dy}{dx}$ is most often used. The derivative of $f(x)$ is usually denoted by $f'(x)$.

Example 1. Find the derivative of cx^2.

Let $y=cx^2$.

Step 1. $y+\delta y=c(x+\delta x)^2=c(x^2+2x\delta x+(\delta x)^2)$

Step 2. $\delta y=c(x+\delta x)^2-cx^2=c(2x\delta x+(\delta x)^2)$

Step 3. $\frac{\delta y}{\delta x}=c(2x+\delta x)$

Step 4. $\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta y}=\lim_{\delta x\to 0}c(2x+\delta x)=2cx.$

Example 2. Find the derivative of $\sqrt{x}$.

Let $y=\sqrt{x}$.

Step 1. $y+\delta y=\sqrt{x+\delta x}$

Step 2. $\delta y=\sqrt{x+\delta x}-\sqrt{x}$

Step 3. $\frac{\delta y}{\delta x}=\frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}$

$$=\frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}\cdot\frac{\sqrt{x+\delta x}+\sqrt{x}}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$=\frac{x+\delta x-x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})}=\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

Step 4. $\dfrac{dy}{dx}=\lim\limits_{\delta x\to 0}\dfrac{1}{\sqrt{x+\delta x}+\sqrt{x}}=\dfrac{1}{2\sqrt{x}}$

Example 3. Find the derivative of x^n, where n is a positive integer.

Let $y=x^n$.

Step 1. $y+\delta y=(x+\delta x)^n$

$$=x^n+nx^{n-1}.\delta x+\frac{n(n-1)}{\underline{|2}}x^{n-2}(\delta x)^2$$

$+\ldots\ldots$(by Binomial Theorem).

Step 2. $\delta y=(x+\delta x)^n-x^n=nx^{n-1}.\delta x$

$$+\frac{n(n-1)}{\underline{|2}}x^{n-2}(\delta x)^2+\text{terms}$$

involving higher powers of δx.

Step 3. $\dfrac{\delta y}{\delta x}=nx^{n-1}+\dfrac{n(n-1)}{2}x^{n-2}\,\delta x+$ terms

involving higher powers of δx.

Step 4. $\dfrac{dy}{dx}=\lim\limits_{\delta x\to 0}\dfrac{\delta y}{\delta x}=n\,x^{n-1}$,

Hence $\dfrac{d}{dx}(x^n)=nx^{n-1}$

Example 4. If n is a rational number then find the derivative of $(ax+b)^n$

Let $y=(ax+b)^n$.

$y+\delta y=[a\,(x+\delta x)+b]^n$

$=(ax+b+a\delta x)^n$

$=(ax+b)^n\left(1+\dfrac{a\delta x}{ax+b}\right)^n$

$$=(ax+b)^n\left\{1+n\frac{a\delta x}{ax+b}+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2+\ldots\ldots\text{terms}\right.$$

$$\left.\text{containing higher powers of }\delta x\right\}.$$

$$\delta y=(ax+b)^n\left\{1+n\frac{a\delta x}{ax+b}+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2+\ldots\ldots\ldots\right\}-(ax+b)^n$$

$$=(ax+b)^n\left\{n\frac{a\delta x}{ax+b}+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2+\ldots\ldots\ldots\ldots\right\}$$

$$\frac{\delta y}{\delta x}=(ax+b)^n\left\{n\frac{a}{ax+b}+\frac{n(n-1)}{2!}\delta x\left(\frac{a}{ax+b}\right)^2+\ldots\ldots\right\}$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=(ax+b)^n\left\{n\frac{a}{ax+b}\right\}$$

$$=na\,(ax+\mathrm{b})^{n-1}.$$

An alternate definition of the derivative of $y=f(x)$ at $x=a$ is

$f'(a)=\lim_{x\to a}\frac{\mathrm{f}(x)-f(a)}{x-a}$ (if limit exists). Using this definition we have for positive integer n ;

At $x=a,\ \frac{d}{dx}(x^n)=\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$ from (*i*) of § 2.7.

In later chapters we will develop formulas which will enable us to find the derivatives with greater ease. However, the formulas themselves will be proved by using the basic definition of the derivative involving the four steps illustrated above. Computing derivative by using these four steps (basic) is called the process of *differentiation from definition*, or *first principles*, or *ab-initio*.

Exercise 3.2

Find the derivative ab-initio w.r.t. x :

(1) x^3 (2) $x^{\frac{3}{2}}$ (3) $x^{-\frac{4}{5}}$

(4) $\frac{1}{x}$ (5) $\frac{1}{x^2}$ (6) $\frac{1}{\sqrt{x}}$

(7) **Show that the definition of the derivative**

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

is in conformity with the original definition given ing 3.3.

ANSWERS

1. $3x^2$
2. $\left(\frac{3}{2}\right)x^{\frac{1}{2}}$
3. $-\frac{4}{5}x^{-\frac{9}{5}}$
4. $-\frac{1}{x^2}$
5. $\frac{-2}{x^3}$
6. $-\frac{1}{2x^{\frac{3}{2}}}$

3.4. *Differentiation.* We now prove some theorems on differentiable functions.

Theorem 3.4.1 :—If $f(x)=c$, c a constant, then

$f'(x)=0$ for all x.

Proof : Let $y=f(x)=c$,

then $y+\delta y=c$

$$\Rightarrow \delta y=0$$

$$\Rightarrow \frac{\delta y}{\delta x}=\frac{0}{\delta x}=0$$

$$\Rightarrow \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=0$$

Hence $\dfrac{dy}{dx}=0$.

Rule 1. The derivative of a constant is zero.

Theorem 3.4.2. If $f(x)=cg(x)$, then $f'(x)=cg'(x)$.

Proof : Let $y=cg(x)$,

then $y+\delta y=cg(x+\delta x)$

$$\Rightarrow \delta y=cg(x+\delta x)-cg(x)$$

$$=c[g(x+\delta x)-g(x)]$$

$$\Rightarrow \frac{\delta y}{\delta x}=c\,\frac{g(x+\delta x)-g(x)]}{\delta x}$$

$$\Rightarrow \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=c\lim_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x}$$

Hence $\frac{dy}{dx} = cg'(x)$.

Rule 2. The derivative of any constant multiple of a function is equal to the product of that constant and derivative of the function.

Theorem 3.4.3. Let $f(x)$ and $g(x)$ be two functions,

then $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$.

Proof : Let $y = f(x) \pm g(x)$.

Then $y + \delta y = f(x+\delta x) \pm g(x+\delta x)$

$$\Rightarrow \delta y = [f(x+\delta x) - f(x)] \pm [g(x+\delta x) - g(x)]$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} \pm \frac{g(x+\delta x) - g(x)}{\delta x}$$

$$\Rightarrow \lim_{\delta x \to 0} \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \pm \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

Hence $\frac{dy}{dx} = f'(x) \pm g'(x)$.

Rule 3. The derivative of the sum or difference of any two functions is the sum or difference of their derivatives.

Cor. A generalized form of this theorem is :

If $y = f_1(x) \pm f_2(x) \pm \ldots\ldots \ldots \pm f_n(x)$, then

$$\frac{dy}{dx} = f_1(x) \pm f_2(x) \pm \ldots\ldots\ldots \pm f_n(x).$$

Theorem 3.4.4. If $f(x)$ and $g(x)$ are two functions

then $\frac{dy}{dx}[f(x).\ g(x)] = f(x).\ \ g'(x) + f(x).g(x)$.

Proof. Let $y = f(x).\ \ g(x) = uv$ (say).

Then $y+\delta y=f(x+\delta x)\,g(x+\delta x)=(u+\delta u)(v+\delta v)$

$$\Rightarrow \delta y=(u+\delta u)(v+\delta v)-uv$$

$$=u\delta v+v\delta u+\delta u\delta v$$

$$\Rightarrow \frac{\delta y}{\delta x}=u\frac{\delta v}{\delta x}+v\frac{\delta u}{\delta x}+\frac{\delta u}{\delta x}\cdot\delta v$$

$$\Rightarrow \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=u\lim_{\delta x\to 0}\frac{\delta v}{\delta x}+v\lim_{\delta x\to 0}\frac{\delta u}{\delta x}+\lim_{\delta x\to 0}\frac{\delta u}{\delta x}\cdot\lim_{\delta x\to 0}\delta v.$$

$$=u\frac{dv}{dx}+v\frac{du}{dx}+\frac{du}{dx}\cdot 0$$

$$=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$=f(x)g'(x)+f'(x)g(x).$$

Rule 4. The derivative of the product of two functions is equal to (the first function) × (derivative of the second).

Plus

(the second function) × (derivative of the first).

Cor. Let u, v, w be functions of x, then

$$(u\,v\,w)'=u'\,v\,w+u\,v\;w+u\,v\,w'$$

Proof. By rule 4, we have $(u\,v\,w)'=(u\,v)w'+(u\,v)'w$

$$=u\,v\,w'+u'\,v\,w+u\,v'\,w$$

$$=u'\,v\,w+u\,v'\,w+u\,v\,w'.$$

Theorem 3.4.5. If u and v are functions of x then,

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$$

Proof. Let $y = \dfrac{u(x)}{v(x)} = \dfrac{u}{v}$ (say).

Then $\quad y+\delta y = \dfrac{u(x+\delta x)}{v(x+\delta x)} = \dfrac{u+\delta u}{v+\delta v}$

$$\Rightarrow \delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{uv+v\delta u-uv-u\delta v}{v(v+\delta v)}$$

$$= \frac{v\delta u - u\delta v}{v(v+\delta v)}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v+\delta v)}$$

$$\Rightarrow \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v+\delta v)}$$

Hence $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Rule 5.—The derivative of the quotient of two functions is equal to—

denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator; all divided by the square of the denominator; *i.e.*,

$$\frac{d}{dx}(\text{quotient}) =$$

$$\frac{(\text{denominator})\dfrac{d}{dx}(\text{numerator}) - (\text{numerator})\dfrac{d}{dx}(\text{denominator})}{(\text{denominator})^2}$$

Theorem 3.4.6. —If n is a rational number, then the derivative of $(ax+b)^n = n(ax+b)^{n-1}a$. Proof is already given in example 4 of §3.3.

Rule 6. *Exponent rule*

$$\frac{d}{dx}[f(x)]^n = n[f(x)]. \quad f(x).$$

Example 1. Given $f(x) = -2x^{16}$, find $f'(x)$.

Solution. By Theorem 3.4.2 with $c = -2$ and $n = 16$ we obtain

$$\begin{aligned} f'(x) &= -2(16)x^{16-1} \\ &= -32x^{15}. \end{aligned}$$

Example 2. Given $f(x) = \frac{2}{x^3}$, find $f'(x)$.

Solution. $f(x) = 2x^{-3}$

$$\begin{aligned} f'(x) &= 2(-3)x^{-3-1} \\ &= -6x^{-4} \\ &= \frac{-6}{x^4}. \end{aligned}$$

Example 3. Find $\frac{dy}{dx}$ if $y = 4x^3 + 5x^2 - 7x + 8$.

Soluion. $\frac{dy}{dx} = 4\frac{d}{dx}(x)^3 + 5\frac{d}{dx}(x)^2 - 7\frac{d}{dx}(x) + \frac{d}{dx}(8)$

$$\begin{aligned} &= 4(3x^2) + 5(2x) - 7(1) + 0 \\ &= 12x^2 + 10x - 7 \end{aligned}$$

Example 4. Find the derivative of

$$\left(x + \frac{1}{x}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

Solution. Let $y = \left(x + \frac{1}{x}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

$$\frac{dy}{dx} = \left(x + \frac{1}{x}\right)\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) + \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

$$\frac{d}{dx}\left(x + \frac{1}{x}\right)$$

$$=\left(x+\frac{1}{x}\right)\left\{\frac{1}{2}x^{-\frac{1}{2}}+\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}\right\}+\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)(1-x^{-2})$$

$$=\tfrac{1}{2}\left(x+\frac{1}{x}\right)\left(\frac{1}{\sqrt{x}}-\frac{1}{x^{\frac{3}{2}}}\right)+\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(1-\frac{1}{x^2}\right).$$

Example 5. Find the derivative of

$$\frac{x^2(x+2)}{(x+1)^3}$$

Solution. Let $y=\dfrac{x^2(x+2)}{(x+1)^3}$

$$\frac{dy}{dx}=\frac{(x+1)^3\frac{d}{dx}x^2(x+2)-x^2(x+2)\frac{d}{dx}(x+1)^3}{[(x+1)^3]^2}$$

$$=\frac{(x+1)^3\{x^2(1)+(x+2)(2x)\}-x^2(x+2).\ 3(x+1)^2}{(x+1)^6}$$

$$=\frac{(x+1)^3(3x^2+4x)-3x^2(x+2)(x+1)^2}{(x+1)^6}$$

$$=\frac{(x+1)^2\{(x+1)(3x^2+4x)-3x^2(x+2)\}}{(x+1)^6}$$

$$=\frac{3x^3+3x^2+4x^2+4x-3x^3-6x^2}{(x+1)^4},\text{ provided } x\neq-1,$$

$$=\frac{x(x+4)}{(x+1)^4}.$$

Example 6. Find the derivative of

$$(ax^3+bx^2+cx+d)^n.$$

Solution. Let $y=(ax^3+bx^2+cx+d)^n$.

Now $f(x)=ax^3+bx^2+cx+d$, so that

$y=\{f(x)\}^n$ whence $\dfrac{dy}{dx}=n\{f(x)\}^{n-1}.f(x)$

or $\dfrac{dy}{dx}=n(ax^3+bx^2+cx+d)^{n-1}(3ax^2+2bx+c).$

Exercise 3.3.

Differentiate the following w.r.t.x.

1. $2x^3-4x^2+3x+1$.

2. $x-2x^3+3x^2-4x+5$.

3. $-3x^3-3x^2+2x-4$.

4. $2x^4-3x^3+2x^2+5x-3$.

5. $2x^2-3x+4-x^{-1}+3x^{-2}$.

6. $10x^4-9x^2-4x^{-5}+7x^{-8}$

7(*a*). $(2x+3)(5x^2-6)$.

(*b*). $(ax^2+b)(cx^2+d)$.

8(*a*). $(x^2+1)(x^3-x^2+2)(x-1)$

(*b*). $(x+2)(x+3)(x+4)$.

9. $\dfrac{2x^4-3x+5}{x^2}$.

10. $\dfrac{2x^3-3x^2+4x-2}{x^3}$.

11. $\dfrac{x-1}{x+1}$.

12. $\dfrac{3x-2}{2x+3}$.

13. $\dfrac{x^2+1}{x^2-1}$.

14. $\dfrac{x}{x^2+1}$.

15. $\dfrac{x+1}{x^2+2x+2}$.

16. $\dfrac{2x^2-3x+4}{x^2-2x+3}.$

17. $\sqrt{x+\sqrt{x}}.$

18. $\sqrt{\dfrac{1+x}{1-x}}$

19. $\dfrac{x\sqrt{a+x}}{\sqrt{a-x}}$

20. $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}.$

21. $\dfrac{\sqrt{x}}{\sqrt{x}+1}$

22. $\dfrac{x}{(a^2+x^2)^{\frac{5}{2}}}.$

23. $(a^2-x^2)^{\frac{2}{3}}.$

24. $(ax^p+bx^q)^r.$

ANSWERS

1. $6x^2-8x+3.$
2. $4x^3-6x^2+6x-4.$
3. $-9x^2-6x+2.$
4. $8x^3-9x^2+4x+5.$
5. $4x-3+\dfrac{1}{x^2}-\dfrac{6}{x^3}.$
6. $10x^4-9x^2-4x^{-5}+7x^{-8}.$

7(*a*). $30x^2+30x-12.$

(*b*). $4acx^3+2x(bc+ad).$

8(*a*). $6x^5-10x^4+8x^3-2x+2.$

(*b*). $3x^2+18x+26.$

9. $4x+\dfrac{3}{x^2}-\dfrac{10}{x^3}.$

10. $\dfrac{3}{x^2}-\dfrac{8}{x^3}+\dfrac{6}{x^4}$

11. $\dfrac{2}{(x+1)^2}.$

12. $\dfrac{13}{(2x+3)^2}.$

13. $\dfrac{-4x}{(x^2-1)^2}.$

14. $\dfrac{1-x^2}{(1+x^2)^2}.$

15. $\dfrac{-x(x+2)}{(x^2+2x+3)^2}.$

16. $\dfrac{-x^2+4x-1}{(x^2-2x+3)^2}.$

17. $\dfrac{1+\dfrac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}$

18. $\dfrac{1}{\sqrt{1-x^2}\,(1-x)}.$

19. $\dfrac{a^2-x^2+ax}{(a+x)^{\frac{1}{2}}\,(a-x)^{\frac{3}{2}}}.$

20. $\dfrac{a}{\sqrt{a^2-x^2}\,(a+\sqrt{a^2-x^2})}$.

21. $\dfrac{1}{2\sqrt{x}\,(\sqrt{x}+1)^2}$.

22. $\dfrac{a^2-4x^2}{(a^2+x^2)^{\frac{7}{2}}}$.

23. $\dfrac{-4x}{3(a^2-x^2)^{\frac{1}{3}}}$.

24. $r(pax^{p-1}+qbx^{q-1})(ax^p+bx^q)^{r-1}$.

3.5. **Composite Functions*. Given two functions say

$$y=u^2 \text{ and } u=5x^2-x$$

we get a new function

$$y=(5x^2-x)^2$$

by putting the values of u from the second function in the first function. Functions constructed in this manner are called composite functions.

Suppose $y=f(u)$ and $u=\varnothing(x)$ are two functions and the range of $\varnothing(x)$ is contained in the domain of $f(u)$.

Then $$y=f(u)=f[\varnothing(x)]$$

defines a composite function in which x is the independent variable and y is the dependent variable. The intermediate variable u has two roles. In $y=f(u)$ it is independent and in $u=\varnothing(x)$ it is dependent.

If we are given $y=f(u)$ and $u=\varnothing(x)$, and if $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$ exist, then it can be shown that $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ (1)

---

*This gives an alternate method of tackling problems, which can otherwise be solved by using Theorem 3.4.6.

This is known as chain rule and sometimes it is proved in the following manner

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x}. \qquad (2)$$

Taking limit as $\delta x \to 0$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

This proof is defective. In order to write (2) we must have $\delta x \neq 0$ and $\delta u \neq 0$. Now in the definition of $\frac{dy}{dx}$ as the limit of $\frac{\delta y}{\delta x}$, δx is the increment of the independent variable and it is required that $\delta x \neq 0$. But there is no guarantee that $\delta u \neq 0$. It can in fact happen that $\delta u = 0$, for $u(x)$ and $u(x+\delta x)$ may have the same value, even when δx is very small.

The proof of (1) can be seen in any book on advanced calculus.

Example 1. If $y=(x^2+3x-2)^4$, find $\frac{dy}{dx}$.

Solution. Let $u=x^2+3x-2$. Then $\frac{du}{dx}=2x+3$.

Since $y=u^4$, $\frac{dy}{du}=4\,u^{4-1}=4\,u^3$.

Hence $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=4\,u^3\,(2x+3)$

$$=4(2x+3)(x^2+3x-2)^3.$$

Example 2. Given $y=(x^3+3x^2-6x+4)^{-1}$, find $\frac{dy}{dx}$.

Solution. Let $u=x^3+3x^2-6x+4$. Then $y=u^{-1}$.

Now $\frac{du}{dx}=3x^2+6x-6$ and $\frac{dy}{du}=-\frac{1}{u^2}$.

$$\text{Hence } \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=-\frac{3x^2+6x-6}{(x^3+3x^2-6x+4)^2}.$$

Example 3. Find $\frac{dy}{dx}$ when $y=\left(\frac{1-x}{1+x}\right)^3$.

Solution. Let $u=\frac{1-x}{1+x}$. Then $y=u^3$.

$$\text{Now } \frac{du}{dx}=\frac{(1+x)(-1)-(1-x).1}{(1+x)^2}$$

$$=\frac{-2}{(1+x)^2}$$

and $\frac{dy}{du}=3u^2$.

$$\text{Hence } \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=3\left[\frac{(1-x)}{(1+x)}\right]^2\cdot\frac{-2}{(1+x)^2}$$

$$=\frac{-6\,(1-x)^2}{(1+x)^4}$$

Example 4. If $y=(a^2+x^2)(a^2-x^2)^{-2}$, find $\frac{dy}{dx}$.

Solution. $y=(a^2+x^2)(a^2-x^2)^{-2}$

$$\frac{dy}{dx}=(a^2+x^2)\frac{d}{dx}(a^2-x^2)^{-2}+(a^2-x^2)^{-2}\frac{d}{dx}(a^2+x^2)$$

$$=(a^2+x^2)(-2)(a^2-x^2)^{-3}(-2x)+(a^2-x^2)^{-2}(2x)$$

$$=2x\left[\frac{2(a^2+x^2)}{(a^2-x^2)^3}+\frac{1}{(a^2-x^2)^2}\right]$$

$$=2x\frac{2a^2+2x^2+a^2-x^2}{(a^2-x^2)^3}$$

$$=\frac{2x(x^2+3a^2)}{(a^2-x^2)^3}.$$

3.6. *Parametric Representation of Curves.* If the coordinates (x, y) of a point P on a curve are given as functions $x=f(u)$, $y=g(u)$ of a third variable or parameter u, the equations $x=f(u)$ $y=g(u)$ are called parametric equations of the curve.

Examples (1). $x=a\sin\theta$, $y=a\cos\theta$ are parametric equations, with parameter θ, of the circle $x^2+y^2=a^2$, since $x^2+y^2=a^2\sin^2\theta+a^2\cos^2\theta=a^2$.

(2) $x=\cos\theta$, $y=4\sin^2\theta$ are parametric equations, with parameter θ of the parabola $4x^2+y=4$, since

$$4x^2+y=4\cos^2\theta+4\sin^2\theta=4.$$

(3). $x=\frac{1}{2}t$, $y=4\,t^2$ is another parametric representation, with parameter t, of the same curve.

(4). $x=a\cos\theta$, $y=b\sin\theta$ are parametric equations, with parameter θ; of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Given parametric equations $x=f(\theta)$, $y=\psi(\theta)$ of a curve, we fiind $\frac{dv}{ax}$ by the formula

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

(5) Given $x=\frac{1}{2}t$, $y=4-t^2$, find $\frac{dy}{dx}$.

Solution. $\frac{dx}{dt}=\frac{1}{2}, \frac{dy}{dt}=-2t.$

$$\text{Hence} \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-2t}{\frac{1}{2}}=-4t.$$

3.7 *Differentiation of Implicit Functions.* Consider the equation $y=2x^2+1.$ (1)

This equation expresses y explicitly as a function of x. Now consider the following equations

$$y^2+x^2y=3 \quad (2)$$

$$x^2+y^2-4=0 \quad (3)$$

$$y^3+x^3-3xy-2=0. \quad (4)$$

These equations show that x and y are related to each other. For any value of x, each of these equations gives one or more corresponding values of y. Thus these equations determine function or functions of x. Equations like these are known as implicit equations and the functions represented by them are called implicit functions.

Two of the implicit functions represented by $x^2+y^2-4=0$ are given by

$$y=\sqrt{4-x^2} \text{ and } y=-\sqrt{4-x^2}$$

Sometimes a relation may be too complicated to be solved for y in terms of x by any simple formula.

We now give examples of differentiation of implicit functions which enable us to find the derivative of such functions.

Example 1. Given $y^3+3xy+x^3=5$, fiind y' in terms of x and y.

Solution. Regarding y as a function of x we differentiate both sides of the equation $y^3+3xy+x^3 = 5$ to get

$$3y^2\frac{dy}{dx}+3x\frac{dy}{dx}+3y+3x^2=0$$

or $$\frac{dy}{dx}=-\frac{x^2+y}{x+y^2}.$$

Exercise 3 4.

1. **Differentiate the following** ·

(*i*) $(2x+3)^{10}$

(*ii*) $(x^2+2x+3)^{\frac{3}{2}}$

(*iii*) $(x^3+3x^2+6x+5)^{\frac{3}{5}}$

(*iv*) $(2x^3-3x^2+3x-1)^{-\frac{1}{3}}$

(*v*) $(x^4+2x^2+1)^{-\frac{7}{4}}$

(*vi*) $\sqrt[3]{(x^3+3x+2)^2}$

(*vii*) $\dfrac{1}{\sqrt{x^2+2x+3}}$

(*viii*) $(2x+3)^4\,(3x-2)^{\frac{7}{3}}$

(*ix*) $(3-2x)^{\frac{5}{2}}(2-3x)^{\frac{1}{3}}$

(*x*) $(2x-1)^{\frac{5}{2}}(7x-3)^{\frac{3}{7}}$

(*xi*) $\sqrt{(1-2x)^3}\,\sqrt[3]{x^2+1}$

2. Differentiate the following functions w. r. t. their independent variable :—

(i) $g(t) = \sqrt[3]{\frac{2+6t}{t}}$

(ii) $\varnothing(s) = \frac{\sqrt{s^2+1}}{s}$

(iii) $f(y) = \frac{2y+3}{\sqrt{y^2+3y+4}}$

(iv) $\varnothing(r) = \frac{\sqrt{1+4r^2}}{r^2}$

(v) $g(s) = \sqrt{2s}+\sqrt{\frac{2}{s}}$

(vi) $k(t) = \frac{2t}{\sqrt{4t^2+3}}$

(vii) $f(x) = \frac{\sqrt{x-2}}{\sqrt[3]{x+2}}$

(viii) $g(x) = \frac{\sqrt{x^2-1}}{\sqrt[3]{x^2+1}}$

(ix) $f(x) = \frac{\sqrt{x-1}}{\sqrt{x+1}}$

(x) $\varnothing(y) = \sqrt[3]{\frac{y^2-1}{y^2+1}}$

(xi) $\varnothing(t) = \sqrt{\frac{3t+4}{3t-2}}$

(xii) $k(x) = \sqrt{\frac{3x-2}{2x+3}}$

(xiii) $f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}$

(*xvi*) $f(x) = \dfrac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}+x}$.

3. Find the value of $f(x)$ for given values of x in the following functions.

(*i*) $f(x)=(x^2+2x+3)^{\frac{2}{3}}$, $x=-1$

(*ii*) $f(x)=\dfrac{2x}{\sqrt{4x^2+1}}$, $x=-1$

4. Find $\dfrac{dy}{dx}$ if

(*i*) $x=2+t,\ y=1+t^2$

(*ii*) $x=t+\dfrac{1}{t},\ y=t+1$

(*iii*) $x=\dfrac{a(1-t^2)}{1+t^2},\ y=\dfrac{2bt}{1+t^2}$

5. If $x=a\theta^2$, $y=2a\theta$, then show that

$$y.\frac{dy}{dx}=2a.$$

6. Find $\dfrac{dy}{dx}$ for the following :

(*i*) $x^2-2xy+2y^2=5$.

(*ii*) $x^3+6xy+5y^3=3$.

(*iii*) $2x^2+3xy+y^2+x+2y-8=0$.

(*iv*) $x^3+2x^2y-xy^2+2y^3=4$.

(*v*) $x^4+2x^2y^2+xy^3+2y^4=6$.

(*vi*) $2x^3-3x^2y+2xy^2-y^3=2$.

(*vii*) $x\sqrt{1+y}+y\sqrt{1+x}=a$.

(*viii*) $x^{\frac{3}{4}}+y^{\frac{3}{4}}=a^{\frac{3}{4}}$.

7. If $x^2+y^2=a^2$, show that

$$\frac{dy}{dx} \quad \frac{dx}{dy}=1.$$

8. If $y=\frac{a}{x^2}$, prove that $x\,\frac{dy}{dx}+2y=0.$

9. If $y=x+\frac{1}{x}$, $x=\frac{1}{\theta-1}$.

Find $\frac{dy}{d\theta}$.

10. If $s=ut+\frac{1}{2}ft^2$, $v=u+ft$, Find $v\frac{dv}{ds}$ if u and are constants.

ANSWERS

1. (*i*) $20(2x+3)^9$

(*ii*) $3(x+1)(x^2+2x+3)^{\frac{1}{2}}$

(*iii*) $\dfrac{2(x^2+2x+2)}{(x^3+3x^2+6x+5)^{\frac{1}{3}}}$

(*iv*) $-\dfrac{(2x^2-2x+1)}{(2x^3-3x^2+3x-1)^{\frac{4}{3}}}$

(*v*) $-\dfrac{7(x^3+x)}{(x^4+3x^2+1)^{\frac{11}{4}}}$

(*vi*) $\dfrac{2(x^2+1)}{(x^3+3x+2)^{\frac{1}{3}}}$

(*vii*) $\dfrac{-(x+1)}{2(x^2+2x+3)^{\frac{5}{4}}}$

(*viii*) $(2x+3)^3(3x-2)^{\frac{4}{3}}(38x+5)$

(*ix*) $(17x-13)(3-2x)^{\frac{3}{2}}(2-3x)^{-\frac{2}{3}}$

(*x*) $(2x-1)^{\frac{3}{2}}(7x-3)^{-\frac{4}{7}}(41x-18)$

(*xi*) $\dfrac{(1-2x)^{\frac{1}{2}}(2x-13x^2-9)}{3(x^2+1)^{\frac{2}{3}}}$

2. (*i*) $-\frac{2}{3}t^{-\frac{4}{3}}(2+6t)^{-\frac{2}{3}}$

(*ii*) $-\dfrac{1}{s^2\sqrt{s^2+1}}$

(*iii*) $\frac{7}{2}(y^2+3y+4)^{-\frac{3}{2}}$

(*iv*) $\dfrac{-2(1+2r^2)}{r^3\sqrt{1+4r^2}}$

(*v*) $\dfrac{s-1}{\sqrt{2}\,s^{\frac{3}{2}}}$

(*vi*) $\dfrac{6}{(4t^2+3)^{\frac{3}{2}}}$

(*vii*) $\dfrac{x+10}{6(x-2)^{\frac{1}{2}}(x+2)^{\frac{4}{3}}}$

(*viii*) $\dfrac{x^3+5x}{3(x^2-1)^{\frac{1}{2}}(x^2+1)^{\frac{4}{3}}}$

(*ix*) $\dfrac{1}{(x+1)\sqrt{x^2-1}}$

(*x*) $\dfrac{4y}{3(y^2-1)^{\frac{2}{3}}(y^2+1)^{\frac{4}{3}}}$

(*xi*) $\dfrac{-6}{(3t-2)^{\frac{4}{3}}(3t+4)^{\frac{2}{3}}}$

(xii) $\dfrac{13}{2(3x-2)^{\frac{1}{2}}(2x+3)^{\frac{3}{2}}}$

(xiii) $\dfrac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}$

(xiv) $-4x+2\left(\dfrac{2x^2-1}{\sqrt{x^2-1}}\right)$

3. (i) 0 (ii) $2.5^{-\frac{3}{2}}$

4. (i) $2t$ (ii) $\dfrac{t^2}{t^2-1}$

(iii) $\dfrac{b(t^2-1)}{2at}$

6. (i) $y'=\dfrac{x-y}{x-2y}$

(ii) $y'=-\left(\dfrac{x^2+2y}{2x+5y^2}\right)$

(iii) $y'=-\left(\dfrac{4x+3y+1}{3x+2y+2}\right)$

(iv) $y'=\dfrac{-3x^2-4xy+y^2}{2x^2-2xy+6y^2}$

(v) $y'=-\left(\dfrac{4x^3+4xy^2+y^3}{4x^2y+3xy^2+8y^3}\right)$

(vi) $y'=\dfrac{6x^2-6xy+2y^2}{3x^2-4xy+3y^2}$

(vii) $-\dfrac{\sqrt{1+y}\,(y+2\sqrt{(1+x)(1+y)})}{\sqrt{1+x}\,(x+2\sqrt{(1+x)(1+y)})}$

(viii) $-\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$

9. $(1-x^2)$ 10. f

3.8. *Antiderivative.*

In the previous sections of this chapter we have been occupied with the problem of finding the derivative of a given function. Many of the most important applications of calculus lead to the inverse problem of finding the function, when its derivative is given. The required function is called the integral or antiderivative of the given derivative, and the process of finding it is called ***integration.***

If $F'(x)=f(x)$ for all x in some interval, then $F(x)$ is said to be an *antiderivative* or an *integral* of $f(x)$ in that interval.

Examples. (*i*) $\frac{1}{4}x^4$ is an antiderivative of x^3.

(*ii*) $\frac{3}{5}x^5-\frac{7}{3}x^3+4x+10$ is an antiderivative of $3x^4-7x^2+4$.

It may be noted that if $F(x)$ is an antiderivative of $f(x)$, so is $F(x)+c$, where c is any constant. This is because the derivative of a constant is zero, and

$$\frac{d}{dx}[F(x)+c]=\frac{d}{dx}F(x)+\frac{d}{dx}c=F'(x)=f(x).$$

If $F(x)$ is a particular antiderivative of $f(x)$, the expression $F(x)+c$, where c is an arbitrary constant, is called the Indefinite integral of $f(x)$, and is denoted by $\int f(x)\,dx$. The function $f(x)$ under the symbol $\int$ is called integrand.

Thus if $F(x)$ is a particular antiderivative of $f(x)$, then $\int f(x)\,dx=F(x)+c$.

Example 1. $\int(3x^2-5x-10)dx=x^3-\frac{5}{2}x^2-10x+c$;

for $\frac{d}{dx}(x^3-\frac{5}{2}x^2-10x+c)=3x^2-5x-10$.

From the fundamental theorems on derivatives and the definition involved, we have :

(*i*) An integral of a constant times a function is the product of the constant and the integral of the function, *i.e.*,

$$\int c\,f(x)dx = c\int f(x)dx.$$

(*ii*) An integral of a sum is the sum of integrals, *i.e.*

$$\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx.$$

Proof. (*i*) We first note by definitions of a derivative and an antiderivative, the symbols $\frac{d}{dx}$ and $\int dx$ cancel each other, *i.e.*, $\frac{d}{dx}\int f(x)dx = f(x)$.

$$\text{Now } \frac{d}{dx}\left\{c\int f(x)dx\right\} = c\frac{d}{dx}\int f(x)dx = c\,f(x).$$

$$\text{Hence } \int\frac{d}{dx}\left\{c\int f(x)dx\right\} = \int c\,f(x)dx$$

$$\Rightarrow c\int f(x)dx \qquad = \int cf(x)dx$$

$$\Rightarrow \int c\,f(x)dx \qquad = c\int f(x)dx.$$

$$(ii)\ \frac{d}{dx}\left\{\int f(x)dx + \int g(x)dx\right\}$$

$$= \frac{d}{dx}\int f(x)dx + \frac{d}{dx}\int g(x)dx$$

$$= f(x)+g(x).$$

$$\text{Hence } \int\frac{d}{dx}\left[\int f(x)dx + g\int(x)dx\right] = \int[f(x)+g(x)]dx.$$

$$\Rightarrow \int f(x)dx + \int g(x)dx = \int[f(x)+g(x)]dx$$

$$\Rightarrow \int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx.$$

3.9. *Formulae for Indefinite Integrals of the form*

$$\int x^n dn, \int (ax+b)^n \, dx.$$

1. Since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$, $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, provided $n \neq 1$

2. Since $\frac{d}{dx}\frac{(ax+b)^n}{a\,(n+1)^n} = (ax+b)^n$,

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} \text{ provided } n \neq -1.$$

From above we conclude that :

Since $\frac{d}{dx}\left(\frac{[f(x)]^{n+1}}{n+1}\right) = [f(x)]^n f'(x)$,

$$\int [f(x)]^n \; f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}; \text{ provided } n \neq -1.$$

Examples : Find

(*i*) $\int (ax^3+bx^2+cx)\, dx.$

(*ii*) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

Solution.

(*i*) $\int (ax^3+bx^2+cx) dx$

$= \int ax^3 dx + \int bx^2 dx + \int cx dx$

$= a\int x^3 dx + b\int x^2 dx + c\int x dx$

$= a\frac{x^4}{4} + b\,\frac{x^3}{3} + c\,\frac{x^2}{2} + C.$

(*ii*) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

$$= \int \frac{\sqrt{x+a} - \sqrt{x+b}}{(x+a)-(x+b)} dx$$

$$= \frac{1}{a-b} \int (\sqrt{x+a} - \sqrt{x+b})\ dx$$

$$= \frac{1}{a-b} \left[\int (x+a)^{\frac{1}{2}}\ dx - \int (x+b)^{\frac{1}{2}}\ dx \right|$$

$$= \frac{1}{a-b} \left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}} \right] + c$$

$$= \frac{2}{3(a-b)} \left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right] + c$$

3.10. *Definite Integral.* An integral evaluated between the end points of an interval is called a definite integral.

The definite integral $\int_a^b f'(x)$ alculated as follows :

$$\int_a^b f(x)\, dx = \Big| f(x)+c \Big|_a^b = \{ f(b)+c \} - \{ f(a)+c \}$$

$= f(b) - f(a)$. The numbers a and b are said to be the limits of this definite integral, where a is referred to as the lower and b as the upper limit.

Example : Evaluate

$$\int_0^1 \frac{x+1}{(x^2+2x+4)^{-\frac{1}{3}}} dx..$$

Solution :

$$\int_0^1 \frac{x+1}{(x^2+2x+4)^{-\frac{1}{3}}} dx$$

$$=\int_0^1 (x^2+2x+4)^{\frac{1}{3}}\ (x+1)\ dx$$

$$=\tfrac{1}{2}\int_0^1 (x^2+2x+4)^{\frac{1}{3}}\ (2x+2)\ dx$$

$$=\tfrac{1}{2}\ \left|\frac{(x^2+2x+4)^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right|_0^1$$

$$=\tfrac{1}{2}\cdot\tfrac{3}{4}\ \Big|(x^2+2x+4)^{\frac{4}{3}}\Big|_0^1$$

$$=\tfrac{3}{8}\left[(1^2+2\ (1)+4)^{\frac{4}{3}}-(0^2+2\ (0)+4)^{\frac{4}{3}}\right]$$

$$=\tfrac{3}{8}\left[7^{\frac{4}{3}}-4^{\frac{4}{3}}\right]=\tfrac{3}{8}\left[\ 7.7^{\ \frac{1}{3}}-4.4^{\frac{1}{3}}\right].$$

Exercise 3.5.

1. Evaluate the following indefinite integrals

(*i*) $\int(x^2-4x+13)\ dx.$

(*ii*) $\int(3x^4-5x^3-4x^2-2)\ dx.$

(*iii*) $\int(4x^3-12x^2-4x+12)\ dx.$

(*iv*) $\int x^2(x^2-4)\ dx.$

(*v*) $\int\sqrt{(2x+5)}\ dx.$

(*vi*) $\int (2x+3)^{\frac{2}{3}}\, dx.$

(*vii*) $\int (3x+4)^{29}\, dx.$

(*viii*) $\int \frac{du}{u^2}.$

(*ix*) $\int \frac{6}{v^5}\, dv.$

(*x*) $\int \frac{dy}{\sqrt{ay+b}}$

(*xi*) $\int x(x^3+1)^2\, dx.$

(*xii*) $\int (x-1)(x-2)(x-3)\, dx.$

2. **Evaluate**

(*i*) $\int (\sqrt{x}-1)^2\, dx.$

(*ii*) $\int_0^2 \frac{dx}{\sqrt{1+x}+\sqrt{x}}$

(*iii*) $\int \frac{x^2+3\sqrt{x}+4}{3x^{\frac{3}{4}}}\, dx.$

(*iv*) $\int \frac{x+8}{\sqrt{x}}\, dx.$

(*v*) $\int \frac{(\sqrt{\theta}-1)^3}{\sqrt{\theta}}\, d\theta.$

(*vi*) $\int \frac{dx}{(2x+3)^{\frac{2}{3}}}$

(*vii*) $\int_{-1}^{1} (2x^2+4)^3 \, (x) \, dx.$

(*viii*) $\int_{0}^{2} (x^2+bx+c)^{-\frac{2}{3}} \, (x+\frac{b}{2}) \, dx.$

(*ix*) $\int \frac{3x^2+2x+1}{(x^3+x^2+x+7)^{\frac{1}{7}}} \, dx.$

ANSWERS

1. (*i*) $\frac{x^3}{3} - 2x^2 + 13x + c$

(*ii*) $\frac{3x^5}{5} - \frac{5x^4}{4} - \frac{4x^3}{3} - 2x + c$

(*iii*) $x^4 - 4x^3 - 2x^2 + 12x + c$

(*iv*) $\frac{x^5}{5} - \frac{4x^3}{3} + c$

(*v*) $\frac{1}{3}(2x+5)^{\frac{3}{2}} + c$

(*vi*) $\frac{3}{10}(2x+3)^{\frac{5}{3}} + c$

(*vii*) $\frac{1}{90}(3x+4)^{30} + c$

(*viii*) $-\frac{1}{u} + c$

(*ix*) $-\frac{3}{2v^4} + c$

(*x*) $\frac{2\sqrt{ay+b}}{a} + c$

(*xi*) $\frac{1}{8}x^8+\frac{2}{5}x^5+\frac{1}{2}x^2+c$

(*xii*) $\frac{1}{4}x^4-2x^3+\frac{11}{2}x^2-6x+c$

2. (*i*) $x+\frac{x^2}{2}-\frac{4}{3}x^{\frac{3}{2}}+c$

(*ii*) $\frac{2}{3}\left[\ 3^{\frac{3}{2}}-2^{\frac{3}{2}}-1\right]$

(*iii*) $\frac{1}{3}\left\{\frac{4}{9}x^{\frac{9}{4}}+4x^{\frac{3}{4}}+16x^{\frac{1}{4}}\right\}+c$

(*iv*) $\frac{2}{3}x^{\frac{3}{2}}+10x^{\frac{1}{3}}+c$

(*v*) $\frac{1}{2}\theta^2-2\theta^{\frac{3}{2}}+3\theta-2\sqrt{\theta}+c$

(*vi*) $\frac{3}{2}(2x+3)^{\frac{1}{3}}+c$

(*vii*) 0

(*viii*) $\frac{3}{2}\{((4+2b+c)^{\frac{1}{3}}-c^{\frac{1}{3}}\}$

(*ix*) $\frac{7}{6}(x^3+x^2+x+7)^{\frac{6}{7}}-c$

Miscellaneous Exercise I

Differentiate ab-initio the following functions.

1. $(ax^2+bx+c)^4$
2. $(t^2+t+1)^{\frac{1}{2}}$

3. $\left(x^{\frac{3}{3}}+\frac{1}{x^{\frac{2}{3}}}\right)^3$

4. $x^{\frac{1}{n}}$

Differentiate the following functions w. r. t. x.

5. $\frac{1}{\sqrt[5]{(1-2x)^3}}$

6. $\frac{(x^2+1)(1-2\sqrt{x})}{x}$

7. $(x-a)(x-b)(x-c)$

8. If $y=\sqrt{x}+\frac{1}{\sqrt{x}}$ show that $2x\frac{dy}{dx}+y=2\sqrt{x}$

9. Find $\frac{dy}{dx}$, when

(*a*) $y=\frac{x}{a-\sqrt{a^2-x^2}}$

(*b*) $y=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$

(*c*) $y=\sqrt{\frac{ax^2+2bx+c}{lx^2+2mx+n}}$

10. Differentiate

(*i*) $\frac{x^2}{x^2-1}$, w. r. t. x^2

Hint : put $x^2=z$, so that $y=\frac{z}{z-1}$. Now find $\frac{dy}{dz}$.

(*ii*) $\frac{1-x^3}{1+x^3}$, w. r. t. x^3

(*iii*) $\frac{x^2+1}{x^2-1}$, w. r. t. x^3 $\left(\text{hint : Put } y=\frac{x^2+1}{x^2-1}, u=x^3 \text{ and find } \frac{dy}{du}\right)$

(*iv*) $x^2-\frac{1}{x^2}$, w. r. t. x^4

(*v*) $2\theta^2+\theta+1$, w. r. t. $\theta^2-\theta-1$

(*vi*) $\frac{ax+b}{cx+d}$, w. r. t. $\frac{ax^2+b}{cx^2+d}$

(*vii*) $\frac{x^2+a^2}{x^2-a^2}$, w. r. t. $\frac{x-a}{x+a}$

11. Find $\frac{dy}{dx}$, if

(*i*) $x=\frac{a_1t^2+a_2t+a_3}{l_1t^2+l_2t+l_3}$, $y=\frac{b_1t^2+b_2t+b_3}{l_1t^2+l_2t+l_3}$

(*ii*) $x=\sqrt{\theta}+\frac{1}{\sqrt{\theta}}$, $y=(\theta^2-2\theta+1)^{\frac{2}{3}}$

(*iii*) $x=\frac{3at}{1+t^3}$, $y=\frac{3at^2}{1+t^3}$

12. Find $\frac{dy}{dx}$, when

(*i*) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(*ii*) $x^2+y^2+2gx+2fy+c=0$

(*iii*) $ax^2\sqrt{y+1}+by^2\sqrt{x+1}=c$

(*iv*) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

(*v*) $\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}=\frac{1}{a}$

13. Integrate the following functions w. r. t. their independent variable.

(*i*) $\sqrt{\theta}-\frac{1}{\sqrt{\theta}}$

(*ii*) $(ax^2+bx+c)^2$

(*iii*) $\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3$

(*iv*) $\left(\frac{1}{y^3}-y\right)$

(*v*) $x^3(x+2)^3(x-1)^2$

(14) Evaluate

(*i*) $\int_{-1}^{1}(x^4+2x^3+3x^2+4x+5)^{\frac{2}{5}}(4+6x+6x^2+4x^3)dx.$

(*ii*) $\int_{0}^{3}\frac{3x^2+1}{(x^3+x+6)^{-\frac{1}{2}}}\,dx$

ANSWERS

1. $4(ax^2+bx+c)^3(2ax+b)$

2. $\frac{2t+1}{2\sqrt{t^2+t+1}}$

3. $2\left(x^{\frac{2}{3}}+\frac{1}{x^{\frac{2}{3}}}\right)^2\left(x^{-\frac{1}{3}}-x^{-\frac{5}{3}}\right)$

4. $\frac{1}{n}x^{\frac{1}{n}-1}$

5. $\frac{6}{5}(1-2x)^{-\frac{8}{5}}$

6. $1+\frac{1}{x^{\frac{3}{2}}}-\frac{1}{x^2}-3\sqrt{x}$

7. $(x-a)(x-b)+(x-a)(x-c)+(x-b)(x-c)$

9. (a) $\frac{-a}{\sqrt{a^2-x^2}(a-\sqrt{a^2-x^2})}$

(b) $\frac{-2}{x^3}\left(1+\frac{1}{\sqrt{1-x^4}}\right)$

(c) $\frac{x^2(am-lb)-x(an-lc)+(bn-mc)}{(lx^2+2mx+n)^{\frac{3}{2}}(ax^2+2bx+c)^{\frac{1}{2}}}$

10. (i) $\frac{-1}{(x^2-1)^2}$

(ii) $\frac{-2}{(1-x^3)^2}$

(iii) $\frac{-4}{3x(x^2-1)^2}$

(iv) $\frac{x^4+1}{2x^6}$

(v) $\frac{4\theta+1}{2\theta-1}$

(vi) $\frac{(cx^2+d)^2}{2x(cx+d)^2}$

(vii) $\frac{-ax}{(x-a)^2}$

11. (i) $\frac{(l_1t^2+l_2t+l_3)(2b_1t+b_2)-(b_1t^2+b_2t+b_3)(2l_1t+l_2)}{(l_1t^2+l_2t+l_3)(2a_1t+a_2)-(a_1t^2+a_2t+a_3)(2l_1t+l_2)}$

(*ii*) $\dfrac{8\,\theta^{\frac{3}{2}}}{3(\theta^2-2\theta+1)^{\frac{1}{3}}}$

(*iii*) $\dfrac{t\,(2-t^3)}{(1-2t^3)}$

12. (*i*) $\dfrac{-b^2x}{a^2y}$

(*ii*) $-\dfrac{x+g}{y+f}$

(*iii*) $-\dfrac{(4ax\sqrt{y+1}\sqrt{x+1}+by^2)2\sqrt{y+1}}{(ax^2+4by\sqrt{x+1}\sqrt{y+1})2\sqrt{x+1}}$

(*iv*) $-\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$

(*v*) $-\left(\dfrac{y}{x}\right)^{\frac{3}{2}}$

13. (*i*) $\frac{2}{3}\theta^{\frac{3}{2}}-2\theta^{\frac{1}{2}}+c$

(*ii*) $\frac{1}{5}a^2x^5+\frac{1}{2}abx^4+\frac{1}{3}(b^2+2ac)x^3+bcx^2+c^2x+d$

(*iii*) $\frac{2}{5}t^{\frac{5}{2}}+2\,t^{\frac{3}{2}}+6\,t^{\frac{1}{2}}-2\,t^{-\frac{1}{2}}+c$

(*iv*) $-\dfrac{1}{2y^2}-\frac{1}{2}\,y^2+c$

(*v*) $\frac{1}{9}x^9+\frac{1}{2}x^8+\frac{1}{7}x^7-\frac{5}{3}x^6-\frac{4}{5}x^5+\frac{1}{2}x^4+c$

14. (*i*) $\dfrac{5}{7}\left\{15^{\frac{7}{5}}-3^{\frac{7}{5}}\right\}$

(*ii*) $2\sqrt{6}(\pm\sqrt{6}-1)$

CHAPTER IV

DIFFERENTIATION OF TRIGONOMETRIC, LOGARITHMIC AND EXPONENTIAL FUNCTIONS

4.1. *Differentiation of Trigonometric functions.*

Theorm 4.1.1. If $f(x) = \text{Sin}\, x$, then $f'(x) = \text{Cos}\, x$

Proof. Let $y = \text{Sin}\, x$.

Then $y + \delta y = \text{Sin}\,(x + \delta x)$

$$\Rightarrow \delta y = \text{Sin}\,(x + \delta x) - \text{Sin}\, x$$

$$= 2\,\text{Cos}\left(x + \frac{\delta x}{2}\right) \cdot \text{Sin}\,\frac{\delta x}{2}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\text{Cos}\left(x + \frac{\delta x}{2}\right) \text{Sin}\,\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$\Rightarrow \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \text{Cos}\left(x + \frac{\delta x}{2}\right) \times \lim_{\delta x \to 0} \frac{\text{Sin}\,\frac{\delta x}{2}}{\frac{\delta x}{2}}.$$

Hence $\dfrac{dy}{dx} = \text{Cos}\, x . 1 \left(\because \lim_{\theta \to 0} \dfrac{\text{Sin}\,\theta}{\theta} = 1\right).$

Thus $\dfrac{d}{dx}(\text{Sin}\, x) = \text{Cos}\, x.$

Corollory. By chain rule $\dfrac{d}{dx}(\text{Sin}\, u) = \text{Cos}\, u \dfrac{du}{dx}.$

Theorem 4.1.2. If $f(x) = \text{Cos}\, x$, then $f'(x) = -\text{Sin}\, x$.

Proof : *First Method (From First Principle)*

Let $y = \text{Cos } x$.

Then $y + \delta y = \text{Cos }(x + \delta x)$

$$\Rightarrow \delta y = \text{Cos }(x + \delta x) - \text{Cos } x$$

$$= -2 \text{ Sin } \left(x + \frac{\delta x}{2}\right) \text{Sin } \frac{\delta x}{2}$$

$$\Rightarrow \frac{\delta y}{\delta x} = -\text{Sin } \left(x + \frac{\delta x}{2}\right) \frac{\text{Sin } \frac{\delta x}{2}}{\frac{\delta x}{2}}.$$

Proceeding to the limit when $\delta x \to 0$, we have

$$\frac{dy}{dx} = -\text{Sin } x$$

$$\therefore \frac{d}{dx}(\text{Cos } x) = -\text{Sin } x.$$

Second Method :

$$y = \text{Cos } x = \text{Sin } \left(\frac{\pi}{2} - x\right).$$

$$\therefore \frac{dy}{dx} = \text{Cos } \left(\frac{\pi}{2} - x\right)(-1) = -\text{Sin } x.$$

Corollory. By chain rule $\frac{d}{dx}(\text{Cos } u) = -\text{Sin } u \frac{du}{dx}$.

Theorem 4.1.3. If $f(x) = \text{Tan } x$, then $f'(x) = \text{Sec}^2 x$.

Proof. *First Method (From First Principle)*

Let $y = \text{Tan } x$.

Then $y + \delta y = \text{Tan }(x + \delta x)$

$$\Rightarrow \delta y = \text{Tan }(x + \delta x) - \text{Tan } x$$

$$= \frac{\text{Sin }(x + \delta x)}{\text{Cos }(x + \delta x)} - \frac{\text{Sin } x}{\text{Cos } x}$$

$$= \frac{\text{Sin }(x + \delta x) \text{ Cos } x - \text{Cos }(x + \delta x) \text{ Sin } x}{\text{Cos }(x + \delta x) \text{ Cos } x}$$

$$=\frac{\text{Sin}\,[(x+\delta x)-x]}{\text{Cos}\,(x+\delta x)\,\text{Cos}\,x}$$

$$=\frac{\text{Sin}\,\delta x}{\text{Cos}\,(x+\delta x)\,\text{Cos}\,x}$$

$$\Rightarrow \frac{\delta y}{\delta x}=\frac{1}{\text{Cos}\,(x+\delta x)\,\text{Cos}\,x}\cdot\frac{\text{Sin}\,\delta x}{\delta x}.$$

Proceeding to limit when $\delta x \to 0$ we have

$$\frac{dy}{dx}=\frac{1}{\text{Cos}^2 x}=\text{Sec}^2\,x.$$

Second Method :

$$y=\text{Tan}\,x=\frac{\text{Sin}\,x}{\text{Cos}\,x}\left(\frac{u}{v}\text{ form}\right)$$

$$\therefore \frac{dy}{dx}=\frac{\text{Cos}\,x\,\text{Cos}\,x-\text{Sin}\,x\,(-\text{Sin}\,x)}{\text{Cos}^2 x}$$

$$=\frac{\text{Cos}^2 x+\text{Sin}^2 x}{\text{Cos}^2 x}$$

$$=\frac{1}{\text{Cos}^2\,x}=\text{Sec}^2 x.$$

Thus $\frac{d}{dx}(\text{Tan}\,x)=\text{Sec}^2 x.$

Corollory. By chain rule $\frac{d}{dx}(\text{Tan}\,u)=\text{Sec}^2 u\,\frac{du}{dx}\cdot$

Theorem 4.1.4. If $f(x)=\text{Cot}\,x$, then
$f'(x)=-\text{Cosec}^2 x.$

Theorem 4.1.5. If $f(x)=\text{Sec}\,x$, then
$f'(x)=\text{Sec}\,x\,\text{Tan}\,x.$

Theorem 4.1.6. If $f(x)=\text{Cosec}\,x$, then
$f'(x)=-\text{Cosec}\,x\,\text{Cot}\,x.$

The proofs of theorems 5.1.4 to 5.1.6 are left to the reader.

Example 1. Differentiate ab-initio $\sqrt{\text{Sec}\,x}.$

Solution Let $y=\sqrt{\operatorname{Sec} x}$,

then $y+\delta y=\sqrt{\operatorname{Sec}(x+\delta x)}$

$$\Rightarrow \delta y=\sqrt{\operatorname{Sec}(x+\delta x)}-\sqrt{\operatorname{Sec} x}$$

$$=\frac{\operatorname{Sec}(x+\delta x)-\operatorname{Sec} x}{\sqrt{\operatorname{Sec}(x+\delta x)}+\sqrt{\operatorname{Sec} x}}\text{(Rationalizing)}$$

$$=\frac{1}{\sqrt{\operatorname{Sec}(x+\delta x)}+\sqrt{\operatorname{Sec} x}}\left[\frac{1}{\operatorname{Cos}(x+\delta x)}-\frac{1}{\operatorname{Cos} x}\right]$$

$$=\frac{1}{\sqrt{\operatorname{Sec}(x+\delta x)}+\sqrt{\operatorname{Sec} x}}\left[\frac{\operatorname{Cos} x-\operatorname{Cos}(x+\delta x)}{\operatorname{Cos}(x+\delta x)\operatorname{Cos} x}\right]$$

$$=\frac{1}{\sqrt{\operatorname{Sec}(x+\delta x)}+\sqrt{\operatorname{Sec} x}}\left[\frac{-2\operatorname{Sin}\left(x+\frac{\delta x}{2}\right)\operatorname{Sin}\frac{\delta x}{2}}{\operatorname{Cos}(x+\delta x)\operatorname{Cos} x}\right]$$

$$\Rightarrow\frac{\delta y}{\delta x}=\frac{1}{\operatorname{Sec}(x+\delta x)+\sqrt{\operatorname{Sec} x}}\left[\frac{-\operatorname{Sin}\left(x+\frac{\delta x}{2}\right).\operatorname{Sin}\frac{\delta x}{2}}{\operatorname{Cos}(x+\delta x)\operatorname{Cos} x.\frac{\delta x}{2}}\right].$$

Taking limit when $\delta x\to 0$, we get

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\operatorname{Sec} x}}\left[\frac{-\operatorname{Sin} x}{\operatorname{Cos} x.\operatorname{Cos} x}\right].1$$

$$=\frac{\operatorname{Sec} x\tan x}{2\sqrt{\operatorname{Sec} x}}.$$

Example 2. Differentiate $\operatorname{Sin}\sqrt{x}$ from first principles

Solution : Let $y=\operatorname{Sin}\sqrt{x}$.

Then $y+\delta y=\operatorname{Sin}\sqrt{x+\delta x}$

or $\delta y=\operatorname{Sin}\sqrt{x+\delta x}-\operatorname{Sin}\sqrt{x}$

$$=2\operatorname{Cos}\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\cdot\operatorname{Sin}\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}$$

$$\text{or } \frac{\delta y}{\delta x}=\frac{2\cos\dfrac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\dfrac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\delta x}$$

$$=\frac{\cos\dfrac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\dfrac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\dfrac{\sqrt{x+\delta x}+\sqrt{x}}{2}\quad 2\,\dfrac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

$[\because\ \delta x=(\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x})]$.

Taking limit when $\delta x\to 0$, so that

$\sqrt{x+\delta x}-\sqrt{x}$ also $\to 0$, we get

$$\frac{dy}{dx}=\frac{1}{2}\ \frac{\dfrac{\cos 2\sqrt{x}}{2}}{\dfrac{2\sqrt{x}}{2}}\ 1=\frac{1}{2\sqrt{x}}\cos\sqrt{x}$$

Example 3. Differenttiate $\operatorname{Tan}^2 x$ by definition.

Solution : Let $y=\operatorname{Tan}^2 x$,

then $y+\delta y=\operatorname{Tan}^2(x+\delta x)$

or $\delta y=\operatorname{Tan}^2(x+\delta x)-\operatorname{Tan}^2 x$

$=[\operatorname{Tan}(x+\delta x)+\tan x][\operatorname{Tan}(x+\delta x)-\operatorname{Tan} x]$

$$=\Big[\operatorname{Tan}(x+\delta x)+\operatorname{Tan} x\Big]\Big[\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\operatorname{Sin} x}{\operatorname{Cos} x}\Big]$$

$$=\Big[\operatorname{Tan}(x+\delta x)+\operatorname{Tan} x\Big]\Big[\frac{\operatorname{Sin}(x+\delta x)\operatorname{Cos} x-\operatorname{Cos}(x+\delta x)\operatorname{Sin} x}{\operatorname{Cos}(x+\delta x)\operatorname{Cos} x}\Big]$$

$$=\Big[\operatorname{Tan}(x+\delta x)+\operatorname{Tan} x\Big]\Big[\frac{\operatorname{Sin}\delta x}{\operatorname{Cos}(x+\delta x)\operatorname{Cos} x}\Big]$$

$$\frac{\delta y}{\delta x}=\left[\text{Tan }(x+\delta x)+\text{Tan }x\right]\left[\frac{\frac{\text{Sin }\delta x}{\delta x}}{\text{Cos }(x+\delta x)\text{ Cos }x}\right]$$

Taking limit when $\delta x\to 0$, we get

$$\frac{dy}{dx}=2\text{ Tan }x.\frac{1}{\text{Cos}^2x}=\text{Tan }x\text{ Sec}^2\,x.$$

Example 4. Given $f\,(x)=\text{Tan }(2x)$, find $f'\,(x)$.

Solution. $f'\,(x)=\text{Sec}^2\,2x\,\frac{d}{dx}\,(2x)=2\text{ Sec}^2 2x.$

Example 5. Given $y=\text{Sin}^3x$ find $\frac{dy}{dx}$.

Solution : $y=(\text{Sin }x)^3$.

$$\frac{dy}{dx}=3\,(\text{Sin }x)^2\frac{d}{dx}\,(\text{Sin }x)=3\text{ Sin}^2x\text{ Cos }x.$$

Example 6. Given $f\,(x)=x^3\text{ Sec}^2 3x$, find $f'\,(x)$.

Solution. $f\,(x)=x^3\,[\text{Sec }(3x)]^2$

$$f'\,(x)=x^3\left[(2\,(\text{Sec }3x)\,.\right]\frac{d}{dx}\left[\text{Sec }3x\right]+\text{Sec}^2 3x.\;3x^2$$

$$=x^3[2\text{ Sec }3x.\;3\text{ Sec }3x\text{ Tan }3x]+3x^2\text{ Sec}^2\,3x.$$

$$=3x^2\text{ Sec}^2 3x\,(2x\text{ Tan }3x+1).$$

Example 7. Find $\frac{d}{dx}\text{ Sin}^4\,(3x^2+2)$.

Solution. $\frac{d}{dx}\text{ Sin}^4\,(3x^2+2)$

$$=4\text{ Sin}^3\,(3x^2+2)\,\frac{d}{dx}\text{ Sin }(3x^2+2)$$

$$=4\text{ Sin}^3\,(3x^2+2)\text{ Cos }(3x^2+2)\frac{d}{dx}\,(3x^2+2)$$

$$=4\text{ Sin}^3\,(3x^2+2)\text{ Cos }(3x^2+2)\;6x$$

$$=24x\text{ Sin}^3(3x^2+2)\text{ Cos }(3x^2+2).$$

Example 8. Find $\frac{dy}{dx}$ if $y = \sqrt{1+\text{Cos}^2 4x.}$

Solution. $$\frac{dy}{dx} = \tfrac{1}{2}(1+\text{Cos}^2 4x)^{-\frac{1}{2}} \frac{d}{dx}(1+\text{Cos}^2 4x)$$

$$= \tfrac{1}{2}(1+\text{Cos}^2 4x)^{-\frac{1}{2}}\ 2\,\text{Cos}\,4x \frac{d}{dx}(\text{Cos}\,4x)$$

$$= \tfrac{1}{2}(1+\text{Cos}^2 4x)^{-\frac{1}{2}}\ 2\,\text{Cos}\,4x\,(-\text{Sin}\,4x)\frac{d}{dx}(4x)$$

$$= \tfrac{1}{2}(1\quad \text{Cos}^2 4x)^{-\frac{1}{2}}\ 2\,\text{Cos}\,4x(-\text{Sin}\,4x).\,4$$

$$= \frac{-4\text{Sin}\,4x\,\text{Cos}\,4x.}{\sqrt{1+\text{Cos}^2 4x}}$$

Example 9. Find $\frac{dy}{dx}$ if $\text{Sin}\,xy+\text{Sec}\,x=2.$

$$\frac{d}{dx}(\text{Sin}\,xy+\text{Sec}\,x) = \frac{d}{dx}(2).$$

$$\frac{d}{dx}\text{Sin}\,xy+\frac{d}{dx}\text{Sec}\,x=0.$$

$$\text{Cos}\,xy\frac{d}{dx}(xy)+\text{Sec}\,x\,\text{Tan}\,x=0.$$

$$\therefore\quad \text{Cos}\,xy\left(x\frac{dy}{dx}+y\right) = -\text{Sec}\,x\,\text{Tan}\,x$$

$$\text{Thus } x\frac{dy}{dx}+y = -\frac{\text{Sec}\,x\,\text{Tan}\,x}{\text{Cos}\,xy}$$

$$\text{and } \frac{dy}{dx} = \frac{\text{Sec}\,x\,\text{Tan}\,x}{x\,\text{Cos}\,xy} - \frac{y}{x}$$

$$= -\left(\frac{\text{Sec}\,x\,\text{Tan}\,x}{x\,\text{Cos}\,xy}+\frac{y}{x}\right)$$

Example 10. If $x=a\,\text{Cos}^3\theta$, $y=b\,\text{Sin}^3\theta$. Find $\frac{dy}{dx}$.

Solution. $\frac{dy}{d\theta}=-3a\,\text{Cos}^2\theta\,\sin\theta,\ \frac{dy}{d\theta}=3b\,\text{Sin}^2\theta\,\text{Cos}\,\theta$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{3b\,\text{Sin}^2\theta\,\text{Cos}\,\theta}{-3a\,\text{Cos}^2\theta\,\text{Sin}\theta}$$

$$-\frac{b}{a}\,\text{Tan}\,\theta.$$

Exercise 4.1

1. Differentiate the following functions from first principle.

(*i*) $\text{Cos}\,2x$ (*ii*) $a\,\text{Sin}\,bx$

(*iii*) $\text{Cos}\sqrt{}$ (*iv*) $\sqrt{\text{Cot}\,x}$

(*v*) Sec^2x (*vi*) $\text{Sin}\,2x+\text{Cos}\,3x$

2. Differentiate the following :

(*i*) $\text{Sin}\,x^4$ (*ii*) $\text{Tan}^n x$

(*iii*) Sec^2x^3 (*iv*) $\text{Cosec}^3\sqrt{x}$

(*v*) $\text{Cot}^4(ax+b)$ (*vi*) Cos^2x^2

3. $\sqrt{\text{Tan}\,(2x+3)}$ 4. $\sqrt{1+\text{Sin}\,3x}$

5. $\text{Cot}\,(ax^2+bx+c)$ 6. $\sqrt{\text{Sin}\sqrt{x}}$

7. $\sqrt{\text{Sec}\,x^2}$ 8. $\sqrt{\text{Cosec}\,(ax+b)}$

9. $\text{Tan}\,(ax+b)^n$ 10. $\text{Cos}^2(\text{Sin}\,x)$

11. $(ax+b)^2\,\text{Cos}\,cx$ 12. $\sqrt{x+1}\,\text{Tan}^3x$

13. $\frac{1+\text{Cos}\,x}{\text{Sin}\,x}$ 14. $\text{Sin}\,(\text{Tan}\,x)$

15. $\text{Sin}^3(x^2+2)$ 16. $\text{Cosec}\,(\text{Cos}\,ax)$

17. $\frac{1-\text{Sin}\,x}{1+\text{Cos}\,x}$ 18. $\sqrt{\frac{1+\text{Tan}\,x}{1-\text{Tan}\,x}}$

19. $\dfrac{(x+1)^{\frac{2}{3}}}{1+\sec x}$ 20. $\text{Tan}\,(\text{Sin}\,x+\text{Cos}\,x)$

21. $\text{Sec}\,(\text{Tan}\,x-\text{Cot}\,x)$. 22. $\text{Sin Sin}\,(\text{Sin}\,x)$.

23 $\text{Cos Tan}\,(\text{Sin}\,x)$ 24. $\text{Sin}\,\text{Cos}^2\sqrt{x}$

25. $\text{Sin}\,mx.\,\text{Sin}^m x$ 26. $\text{Cot}\,\dfrac{1+x^2}{1+x^3}$

27. $\text{Sin}\sqrt{\dfrac{1+2x}{1+x}}$ 28. $\text{Cos}\left(\text{Cot}\dfrac{1}{x}\right)$

29. Find $\dfrac{dy}{dx}$, when

(*i*) $x=a\,\text{Sin}\,t,\ \ y=\text{Cos}\,at$
(*ii*) $x=a\,\text{Cos}^n\theta,\ y=b\,\text{Sin}^n\theta$
(*iii*) $x=a\,(\text{Cos}\,t+\text{Sin}\,t),\ y=a\,(\text{Sin}\,t-t\,\text{Cos}\,t)$
(*iv*) $x=2t^2+t+1,\qquad y=(\text{Sin}\,t+\text{Cos}\,t)$

30. Differentiate

(*i*) Sin x w.r.t Sin x ·
(*ii*) Cos x w.r.t. Tan x
(*iii*) Tan^2x w.r.t. Cot^3x
(*iv*) Cosec $(ax+b)$ w.r.t. $(ax+b)$

31. If $\text{Sin}\,y=x\,\text{Sin}\,(a+y)$, show that

$$\frac{dy}{dx}=\frac{\text{Sin}^2(a+y)}{\text{Sin}\,a}$$

32. If $yx=\text{Cos}\,(x+y)$, show that

$$\frac{dy}{dx}+\frac{y+\text{Sin}\,(x+y)}{x+\text{Sin}(x+y)}=0$$

33. If $y=\sqrt{\text{Cos}x+\sqrt{\text{Cos}x+\sqrt{\text{Cos}x+\ldots\ldots\infty}}}$, Prove that $\dfrac{dy}{dx}(2y-1)+\text{Sin}\,x=0$.

Solution. $y=\sqrt{\text{Cos } x+\sqrt{\text{Cos } x+\sqrt{\text{Cos } x+\ldots\ldots\ldots\infty}}}$

$$\Rightarrow y=\sqrt{\text{Cos } x+y}$$

$$\frac{dy}{dx}=\frac{-\text{Sin } x+\frac{dy}{dx}}{2\sqrt{\text{Cos } x+y}}=\frac{-\text{Sin } x+\frac{dy}{dx}}{2y}$$

or $\quad \frac{dy}{dx}\left(2y-1\right)+\text{Sin } x=0$

34. If $y=\sqrt{\text{Tan } x+\sqrt{\text{Tan } x+\sqrt{\text{Tan } x+\ldots\ldots\infty}}}$, Prove that $(2y-1)\frac{dy}{dx}=\text{Sec}^2x$.

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1. (*i*) $-2 \text{ Sin } 2x$
   (*ii*) $ab \text{ Cos } bx$
   (*iii*) $-\frac{1}{2\sqrt{x}}\text{Sin }\sqrt{x}$
   (*iv*) $\frac{\text{Cosec}^2x}{2\sqrt{\text{Cot } x}}$
   (*v*) $2 \text{ Sec}^2 x \text{ Tan } x$
   (*vi*) $2 \text{ Cos } 2x-3 \text{ Sin } 3x$
2. (*i*) $4x^3 \text{ Cos } x^4$
   (*ii*) $n \text{ Tan}^{n-1}x \text{ Sec}^2 x$
   (*iii*) $6x^2 \text{Sec}^2 x^3 \text{ Tan } x^3$
   (*iv*) $-\frac{3 \text{ Cosec}^3\sqrt{x} \text{ Cot}\sqrt{x}}{2\sqrt{x}}$
   (*v*) $-4a \text{ Cot}^3 (ax+b) \text{ Cosec}^2 (ax+b)$
   (*vi*) $-4x \text{ Cos } x^2 \text{ Sin } x^2$

3. $\dfrac{\text{Sec}^2(2x+3)}{\sqrt{\text{Tan}(2x+3)}}$

4. $\dfrac{3\text{ Cos }3x}{2\sqrt{1+\text{Sin }3x}}$

5. $-(2ax+b)\text{ Cosec}^2(ax^2+bx+c)$

6 $\dfrac{\text{Cos }\sqrt{x}}{4\sqrt{x}\sqrt{\text{Sin}\sqrt{x}}}$

7. $\dfrac{x\text{ Sec}x^2\text{ Tan }x^2}{\sqrt{\text{Sec }x^2}}$

8. $\dfrac{-a\text{ Cosec }(ax+b)\text{ Cot }(ax+b)}{2\sqrt{\text{Cosec }(ax+b)}}$

9. $na\,(ax+b)^{n-1}\text{ Sec}^2(ax+b)^n$

10. $-2\text{ Cos (Sin }x)\text{ Sin (Sin }x)\text{ Cos }x$

11. $2a\,(ax+b)\text{ Cos }cx-c\,(ax+b)^2\text{ Sin }cx$

12. $\sqrt{x+1}\;3\text{ Tan}^2x\text{ Sec}^2x+\dfrac{1}{2\sqrt{x+1}}\text{ Tan}^3x.$

13. $-\dfrac{1}{2}\text{ Cosec}^2\dfrac{x}{2}$

14. $\text{Cos (Tan }x).\text{ Sec}^2x$

15. $6x\text{ Sin}^2(x^2+2)\text{ Cos }(x^2+2)$

16. $a\text{ Sin }ax.\text{ Cosec (Cos }ax)\text{ Cot (Cos }ax)$

17. $\dfrac{\text{Sin }x-\text{Cos }x-1}{(1+\text{Cos }x)^2}$

18. $\dfrac{1+\text{Tan}^2x}{\sqrt{1+\text{Tan }x}\,(1-\text{Tan }x)^{\frac{3}{2}}}$

19. $\dfrac{2(1+\text{Sec } x)-3\,(x+1)\text{ Sec } x \text{ Tan } x}{3\,(1+\text{Sec } x)^2\,(x+1)^{\frac{1}{3}}}$

20. $(\text{Cos } x-\text{Sin } x)\text{ Sec}^2\,(\text{Sin } x+\text{Cos } x)$

21. $\text{Sec }(\text{Tan } x-\text{Cot } x)\text{ Tan }(\text{Tan } x-\text{Cot } x)\,(\text{Sec}^2 x$
$+\text{Cosee}^2 x)$

22. $\text{Cos Sin }(\text{Sin } x).\ \text{Cos }(\text{Sin } x).\ \text{Cos } x$

23. $-\text{Sin Tan }(\text{Sin } x).\ \text{Sec}^2\,(\text{Sin } x)\ .\ \text{Cos } x$

24. $\dfrac{-1}{\sqrt{x}}\text{Cos}^2\sqrt{x}.\ \text{Cos}\sqrt{x}.\quad \text{Sin}\sqrt{x}\ .$

25. $m\text{ Sin } mx\text{ Sin}^{m-1} x\text{ Cos } x+m\text{ Cos } mx\text{ Sin}^m x$

26 $\dfrac{x^4+3x^2-2x}{(1+x^3)^2}\text{cosec}^2\ \dfrac{1+x^2}{1+x^3}$

27. $\dfrac{1}{2\,(1+x)^{\frac{2}{2}}\sqrt{1+2x}}\ \text{cos}\ \sqrt{\dfrac{1+2x}{1+x}}$

28. $-\dfrac{1}{x^2}\text{ Sin }\left(\text{Cot}\dfrac{1}{x}\right).\ \text{Cosec}^2\dfrac{1}{x}$

29. (*i*) $-\dfrac{\text{Sin } at}{\text{Cos } t}$

(*ii*) $-\dfrac{b}{a}\text{Tan}^{n-2}\,\theta$

(*iii*) $-\dfrac{t\text{ Sin } t}{\text{Cos } t-\text{Sin } t}$

(*iv*) $\dfrac{\text{Cos } t-\text{Sin } t}{(4t+1)}$

30. (*i*) 1

(*ii*) $-\text{Sin } x\text{ Cos}^2 x$

(*iii*) $-\dfrac{2\text{ Tan } x\text{ Sec}^2 x}{3\text{ Cot}^2 x\text{ Cosec}^2 x}$

(*iv*) $-\text{Cosec }(ax+b)\text{ Cot }(ax+b)$

4.2. *Differentiation of Inverse Trigonometric Functions.*

Theorem 4.2.1. If $y = \text{arc Sin } x$, then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \text{ if } -1 < x < 1.$$

(Recall tha we define arc Sin x as the number y for which $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ and $\text{Sin } y = x$)

Proof : Let $y = \text{arc Sin } x$, so that $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ and

$$x = \text{Sin } y.$$

Differentiating w.r.t. x we see that

$$1 = \cos y \frac{dy}{dx} \text{ or } \frac{dy}{dx} = \frac{1}{\cos y}.$$

In order the denominator may not be zero, let us take $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Since $\text{Cos}^2 y = 1 - \text{Sin}^2 y = 1 - x^2$, $\text{Cos } y = \pm\sqrt{1-x^2}$. But for values of y lying in the range $-\frac{\pi}{2} < y < \frac{\pi}{2}$ $\text{Cos } y > 0$.

Thus $\text{Cos } y = \sqrt{1-x^2}$. Hence $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$,

$$-1 < x < 1 \text{ i.e. } (\text{Sin}^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

Cor. 1. Using chain rule $\frac{d}{dx} \text{Sin}^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

Theorem 4.2.2. If $y = \text{arc Cos } x$, then

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}, \ -1 < x < 1.$$

Theorem 4.2.3. If $y = \text{arc Tan } x$, then

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

Theorem 4.2.4. If $y = \text{arc Cot } x$, then

$$\frac{dy}{dx} = \frac{-1}{1+x^2}.$$

Theorem 4.2.5. If $y = \text{arc Sec } x$, then

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}, (\mid x \mid > 1).$$

Theorem 4.2.6. If $y = \text{arc Cosec } x$ then

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}, (\mid x \mid > 1).$$

The proofs of the theorems 4. 2. 2. to 4.2.6. are left as exercises.

Example 1. Given $y = \text{arc Sin}\left(\frac{x}{a}\right)$, find $\frac{dy}{dx}$

(a being a constant >0).

Solution. $$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= \frac{1}{\sqrt{a^2-x^2}}.$$

Example 2. Given $f(x) = x \text{ Tan}^{-1}(x^2)$, find $f'(x)$.

Solution. $$f'(x) = 1. \text{ Tan}^{-1}(x^2) + x\,\frac{1}{1+(x^2)^2}\frac{d}{dx}(x^2)$$

$$= \text{Tan}^{-1}(x^2) + \frac{x}{1+x^4}(2x)$$

$$= \text{Tan}^{-1} x^2 + \frac{2x^2}{1+x^4}.$$

Exercise 4.2

1. Prove theorems 4.2.2 to 4.2.6.

2. Differentiate w.r.t. x :

(*i*) $\text{Sin}^{-1}\dfrac{x}{3}$

(*ii*) $\text{Tan}^{-1}\dfrac{4x-3}{\sqrt{2}}$

(*iii*) $\text{Cos}^{-1}\dfrac{1}{x^2}$

(*iv*) $\text{Sin}^{-1}x^2$

(*v*) $\text{Sec}^{-1}\left(\dfrac{1}{x}+x\right)$

(*vi*) $\text{Cos}^{-1}3x$

(*vii*) $\text{Cot}^{-1}\dfrac{1}{x}$

(*viii*) $\text{Cosec}^{-1}\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)$

3. **Find** y' **if**

(*i*) $y=\text{Sin}^{-1}\sqrt{x}$

(*ii*) $y=\text{Cos}^{-1}\dfrac{x-2}{2}$

(*iii*) $y=\text{Cos}^{-1}\dfrac{8x-9}{9}$

(*iv*) $y=\text{Cot}^{-1}\dfrac{2x}{1-x^2}$

(*v*) $y=\text{Tan}^{-1}\dfrac{x-1}{x+1}$

(*vi*) $y=\text{Sin}^{-1}\dfrac{2x+1}{\sqrt{3}}$

4. If $y = \text{Tan}\left(2\ \text{Tan}^{-1}\frac{x}{2}\right)$, show that $\frac{dy}{dx} = 4\ \frac{1+y^2}{4+x^2}$

5. If $y = x\text{Tan}^{-1}\frac{x}{y}$, show that $\frac{dy}{dx} = \frac{y}{x}$

6. If $y = \text{Sec}^{-1}\left(\frac{x^2+1}{x^2-1}\right)$, prove that $\frac{dy}{dx} = \frac{-2}{x^2+1}$

7. Show that $\frac{d}{dx}\ \text{Sin}^{-1}\frac{1}{x} = \frac{1}{|x|\ \sqrt{x^2-1}}$

$$\frac{d}{dx}\ \text{Cos}^{-1}\frac{1}{x} = \frac{1}{|x|\sqrt{x^2-1}}$$

8. Find
(i) $\frac{d}{dx}\text{Cos}^{-1}\frac{1-x^2}{1+x^2}$

(ii) $\frac{d}{dx}\ \text{Sin}^{-1}\sqrt{1-x^2}$

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2. (i) $\frac{1}{\sqrt{9-x^2}}$

(ii) $\frac{4\sqrt{2}}{16x^2-24x+11}$

(iii) $\frac{2}{x\sqrt{x^4-1}}$

(iv) $\frac{2x}{\sqrt{1-x^4}}$

(v) $\frac{x^2-1}{(x^3+x)\sqrt{\left(x+\frac{1}{x}\right)^2-1}}$

(*vi*) $\frac{-3}{\sqrt{1-9x^2}}$

(*vii*) $\frac{1}{x^2+1}$

(*viii*) $\frac{1-x}{2x\,(x+1)} \times \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 - 1$

3. (*i*) $\frac{1}{2\sqrt{x-x^2}}$

(*ii*) $\frac{-1}{\sqrt{4x-x^2}}$

(*iii*) $\frac{-2}{\sqrt{9x-4x^2}}$

(*iv*) $\frac{-2}{1+x^2}$

(*v*) $\frac{1}{1+x^2}$

(*vi*) $\frac{2}{\sqrt{2-4x-4x^2}}$

8. (*i*) $\frac{2}{1+x^2}$

(*ii*) $\frac{-1}{\sqrt{1-x^2}}$

4.3. *Exponential Functions.* If $a>0$, the exponential function with base a is defined as $f(x)=a^x$.

The students are expected to know the following results :

(*i*) $a^x a^y = a^{x+y}$

(*ii*) $\frac{a^x}{a^y} = a^{x-y}$

(*iii*) $a^x . b^x = (ab)^x$

(*iv*) $(a^x)^y = a^{xy}$

(*v*) $a^o = 1$

(*vi*) If $a<1$, $\lim\limits_{x\to\infty} a^x = 0$, $\lim\limits_{x\to-\infty} a^x = \infty$

(*viii*) If $a>1$ $\lim\limits_{x\to\infty} a^a = \infty$, $\lim\limits_{x\to-\infty} a^o = 0$

Example. Sketch the graph of the function $f(x) = 2^x$.

Solution. The graph is sketched below with the help of the following table :

x	2^x
$-\infty$	0
-3	1/8
-2	1/4
-1	1/2
0	1
1	2
2	4
3	8
4	16

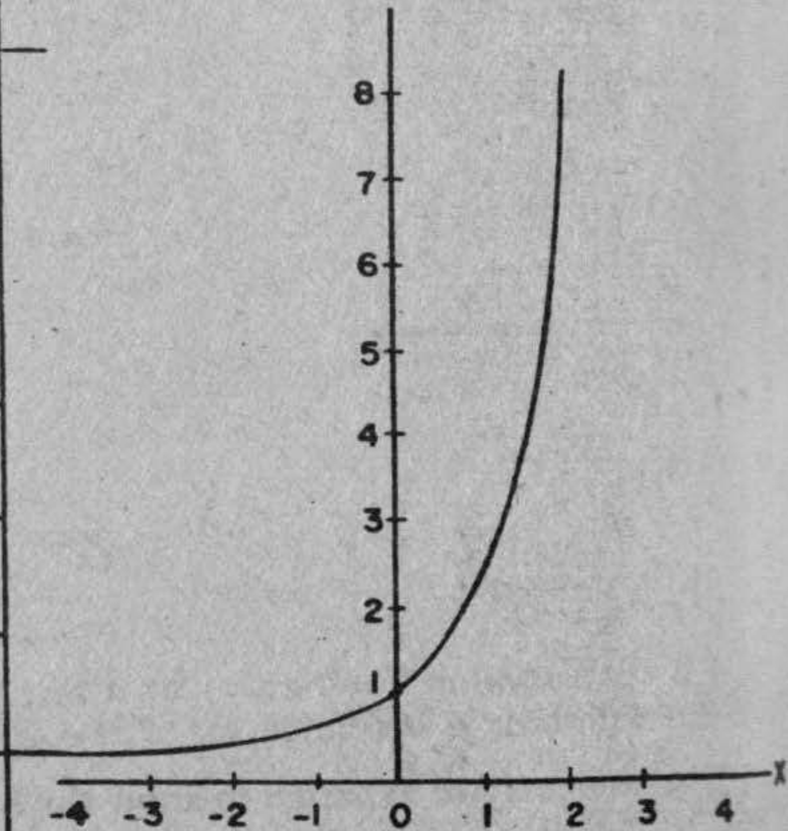

Fig. 4.1

Exercise 4.3

1. Guess the approximate value of $(0.1)^{0\cdot1}$.

Sketch the graph of the following functions :

2. $g(x)=2^{-x}$

3. $h(x)=2^{1+x}$

4. $F(x)=3^{1-x}$

4.4. *The Number e.* Consider the function $f(x)$ given by the infinite series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\cdots\cdots\cdots\cdots(1)$$

For $x=1$, the value of the above function is.

$$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots\cdots\cdots\cdots(2).$$

We use the symbol e to denote the sum of the series (2). Now we show that e lies between 2 and 3. The first two terms of the series (2) show that $e>2$.

Also

$$e=1+1+\frac{1}{2}+\frac{1}{3\cdot2}+\frac{1}{4.3.2}+\cdots\cdots$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots\cdots\cdots$$

$<1+$a G. P. whose sum is 2
<3.

The number e thus lies between 2 and 3.

To find a better approximate value of e, correct to, say 3 decimal places we evaluate a few terms in the beginning of the series (2).

$$e=1+1+0\cdot5+0\cdot16667+0\cdot4167+0\cdot00833+0\cdot00139$$
$$+0\cdot002+0\cdot00002+\cdots\cdots\cdots$$
$$=2\cdot718 \text{ approximately.}$$

It can be shown that e is an irrational number, the proof being beyond the scope of this book. A rational approximation of e to ten decimal places is 2·7182818258.

4·5. *The Function e^x and its Derivative.* It can be shown* that, for all values of x,

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots\ldots$$

The function e^x is called *The Exponential Function*, and is also written as exp (x). In future whenever we speak of the exponential functions, we will assume the base to be the number e unless otherwise stated.

Assuming that the above series can be differentiated term by term, we have

$$\frac{d}{dx}(e^x)=0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{4x^3}{4!}+\ldots\ldots$$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots$$

$$=e^x.$$

Thus if $y=e^x$, then $\dfrac{dy}{dx}=e^x=y$.

The exponential function has thus the property that its derivative *i.e* (rate of change) is equal to itself.

Just as the number π is expectd to make its appearance whenever we make calculations involving circles or allied curves, in the same manner we expect the exponential function to be in the study of those natural phenomena where the rate of ehange of a quantity is proportional to the quantity itself. Examples of the latter are the bacterial growth, decomposition of a radioactive substance, and motion when resistance is proportional to the velocity.

---

* The students who are interested in the proof may refer to § 4 6·

Example 1. Find the derivative of e^{ax} w. r. t . x

Solution. Take $z = ax$, then $e^{ax} = e^{z}, \frac{dz}{dx} = a$

$$\therefore \frac{d}{dx}(e^{ax}) = \frac{d}{dz}(e^{z}) . \frac{dz}{dx} = a . e^{ax}.$$

Example 2. Differentiate $y = e^{x^2+2x}$.

Solution. Take $z = x^2 + 2x$, so that $y = e^{z}$. Now

$$\frac{dz}{dx} = 2x + 2$$

$$\frac{dy}{dz} = e^{z} = e^{x^2+2x}. \text{ But } \frac{dy}{dx} = \frac{dy}{dz} . \frac{dz}{dx}$$

$$= (2x + 2)\, e^{x^2+2x}$$

$$= 2(x1)\, e^{x^2+2x}.$$

Exercise 4.4

1. Sketch the graph of the following functions :

(i) $y = e^{x}$ (ii) $y = e^{-x}$

(iii) $y = \frac{1}{2}(e^{x} + e^{-x})$ (iv) $\frac{1}{2}(e^{x} - e^{-x})$

2. Find the derivatives of

(i) xe^{x} (ii) e^{x} Cos x

(iii) e^{ax} Sin bx (iv) $e^{\text{Sin}\, x}$ Cos x

(v) $e^{x - \frac{x^2}{2}}$ (vi) Sin(e^{x})

(vii) Sin$^{-1}(e^{x})$ (viii) $\frac{e^{2x} - 1}{e^{2x} + 1}$

ANSWERS

2. (*i*) $e^x + xe^x$

(*ii*) $e^x\,(\text{Cos}\,x - \text{Sin}\,x)$

(*iii*) $e^{ax}\,(a\,\text{Sin}\,bx + b\,\text{Cos}\,bx)$

(*iv*) $e^{\text{Sin}\,x}\,(\text{Cos}^2x - \text{Sin}\,x)$

(*v*) $(1-x)e^{x-\frac{x^2}{2}}$

(*vi*) $e^x\,\text{Cos}\,e^x$

(*vii*) $\dfrac{-e^{-x}}{\sqrt{1-e^{-2x}}}$

(*viii*) $\dfrac{4e^{2x}}{(e^{2x}+1)^2}$

4.6*. *e As A Limit.*

(*i*) Consider the expression $\left(1+\dfrac{1}{n}\right)^n$. Expanding this with the help of Binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n = 1 + n.\frac{1}{n} + \frac{n\,(n-1)}{2\,!}\left(\frac{1}{n}\right)^2 + \frac{n\,(n-1)\,(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \ldots\ldots\ldots$$

$$= 1 + 1 + \frac{n\,(n-1)}{n^2}\cdot\frac{1}{2!} + \frac{n\,(n-1)\,(n-2)}{n^3}\cdot\frac{1}{3!} + \cdots$$

$$= 1 + 1 + \frac{1-\frac{1}{n}}{2\,!} + \frac{(1-\frac{1}{n})\,(1-\frac{2}{n})}{3\,!} + \ldots\ldots\ldots$$

Taking the limit when $n \to \infty$

$$\underset{n\to\infty}{\text{limit}}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

---

* The students may omit the proof of this section.

We recognize the series on the right to be the number e.

Therefore $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

(*ii*) Now consider the expansion

$$\left(1+\frac{x}{n}\right)^n = 1+n\left(\frac{x}{n}\right)+\frac{n\,(n-1)}{2!}\left(\frac{x}{n}\right)^2+\ldots\ldots\ldots\ldots$$

$$=1+x+\left(1-\frac{1}{n}\right).\ \frac{x^2}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{x^3}{3!}+\cdots$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots$$

$$\text{Also} \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{z\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}\cdot x} = \left[\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right]^x$$

$$=\left[\left(1+\frac{1}{z}\right)^z\right]^x, \text{taking } z=\frac{n}{x}$$

and so $z\to\infty$ as $n\to\infty$.

$$\therefore \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{z\to\infty}\left[\left(1+\frac{1}{n}\right)^z\right]^x$$

$$=\left[\lim_{z\to\infty}\left(1+\frac{1}{z}\right)^z\right]^x = e^x.$$

Hence $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots\ldots$

4.7. ***The Logarithmic Function.*** **For a positive number a, if $a^m = x$, then we say that $\log_a x = m$.**

It is clear that domain of log x is the set of all the positive real numbers and that the range is the set of all real numbers.

1. **We already know that for all x, $y > 0$,**

(i) $\log xy = \log x + \log y$

(*ii*) $\log \frac{x}{y} = \log x - \log y$

(*iii*) $\log x^n = n \log x$

(*iv*) $\log_a x = \log_b x . \log_a b$

We recall that we have been using tables of logarithms to the base 10, called common logarithms. The base 10 was selected because we use the decimal system in arithmatic, and we have ease in computing the common logarithm of a number in terms of its characteristic and mantissa. Another very important base for logarithm is the number e and $\log_e x$ is called the *Natural Logarithm* ; $\log_e x$ is usually written as lnx. Consider the function $y = lnx = \log_e x$,

$$\therefore \quad x = e^y.$$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = e^y = x.$$

Hence $\frac{d}{dx}(lnx) = \frac{1}{x}$.

Example 1. Find the derivative of $\log(x^2 - 3x + 2)$.

Solution. Let $z = x^2 - 3x + 2$, then $\frac{dz}{dx} = 2x - 3$

$$\frac{d}{dx}\log(x^2 - 3x + 2) = \frac{d}{dx}\left(\log z\right)\frac{dz}{dx}$$

$$= \frac{1}{z}(2x - 3) = \frac{2x - 3}{x^2 - 3x + 2}.$$

Example 2. Differentiate $\log\sqrt{\frac{x^2 + 1}{x^2 - 1}}$.

Solution. $\log\sqrt{\frac{x^2 + 1}{x^2 - 1}} = \frac{1}{2}\log x^2 + 1) - \frac{1}{2}\log(x^2 - 1)$

$$\therefore \quad \frac{d}{dx}\left(\sqrt{\frac{x^2+1}{x^2-1}}\right) = \tfrac{1}{2}\,\frac{d}{dx}\Big[\log(x^2+1)\Big] - \tfrac{1}{2}\cdot\frac{d}{dx}\Big[\log(x^2-1)\Big]$$

$$= \tfrac{1}{2}\cdot\frac{2x}{x^2+1} - \tfrac{1}{2}\,\frac{2x}{x^2-1} = \frac{x}{x^2+1} - \frac{x}{x^2-1} = \frac{2x}{x^4-1}.$$

Example 3. Differentiate $\log_a x$.

Solution. Using (*iv*) for change of base,

$$\log_a x = \log_e x.\ \log_a e = \frac{1}{\log_e a}\cdot \log_e x$$

$$\therefore \quad \frac{d}{dx}(\log_a x) = \frac{1}{\log_e a}\cdot \frac{d}{dx}(\log_e x) = \frac{1}{x\log_e a} = \frac{1}{x ln a}.$$

4.8. *Logarithmic Differentiation.* In some problems the work of differentiation is simplified if we first make use of the laws of logarithms.

Example 1. Differentiate $\dfrac{x^2\sqrt{x^2-1}}{(x+3)^2}$ w.r.t. x.

Solution. Let $y = \dfrac{x^2\sqrt{x^2-1}}{(x+3)^2}$

Taking logarithms of both sides we get

$$lny = 2lnx + \tfrac{1}{2}\,ln\,(x^2-1) - ln\,(x+3)$$

Differentiating both sides w.r.t x.

$$\frac{1}{y}\,\frac{dy}{dx} = 2.\,\frac{1}{x} + \tfrac{1}{2}\,\frac{2x}{x^2-1} - \frac{2}{x+3} = \frac{x^3-9x^2-6}{x(x^2-1)(x+3)}$$

$$\therefore \frac{dy}{dx} = y\,\frac{x^3-9x^2-6}{x(x^2-1)(x+3)} = \frac{x^2\sqrt{x^2-1}}{(x+3)^2}\cdot\frac{x^3+9x^2-6}{x(x^2-1)(x+3)}$$

$$= \frac{x^4+9x^3-6x}{(x^2-1)^{\frac{1}{2}}(x+3)^3}$$

Example 2. Differentiate a^x w.r.t. x.

Eolution. Let $y = a^x$

$$\therefore \quad lny = xlna.$$

Differntiating both sides w.r.t. x

$$\frac{1}{y} \cdot \frac{dy}{dx} = lna$$

$$\therefore \frac{dy}{dx} = y \; lna = a^x lna.$$

We note that $\int a^x dx = \frac{a^x}{lna} + c.$

Example 3. By taking logarithms, find the derivative of x^n for any number n.

Solution. $y = x^n$

$$\Rightarrow lny = n \; lnx$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{n}{x}$$

$$\therefore \quad \frac{dy}{dx} = y . \frac{n}{x} = nx^{n-1}$$

Example 4. Differentiate $x^{\sin x}$

Solution. Let $y = x^{\sin x}$

Then $lny = \sin x \; lnx$

$$\frac{1}{y} \frac{dy}{dx} = \frac{\text{Sin}\, x}{x} + \text{Cos}\, x \; lnx,$$

So that $\frac{dy}{dx} = y\left(\frac{\text{Sin}\, x}{x} + \text{Cos}\, x \; lnx\right)$

$$= x^{\sin x}\left(\frac{\text{Sin}\, x}{x} + \text{Cos}\, x \; lnx\right).$$

Exercise 4.5

1. Sketch the graph of the function $\log x$, $x > 0$.
2. Find the derivatives of :
   (i) $\log_r x$

(*ii*) $x \ln x$

(*iii*) $x \ln x - x$

(*iv*) $\frac{1}{x} \ln x$

(*v*) $\frac{x}{\ln x}$

(*vi*) $e^x \ln x$

(*vii*) $\ln (e^{x+2})$

(*viii*) $\ln \ln x$

(*ix*) $\ln \operatorname{Sin} x$

(*x*) $\ln \operatorname{Cos} \frac{1}{x}$

(*xi*) $\ln (\operatorname{Sec} x + \operatorname{Tan} x)$

(*xii*) $\ln \operatorname{Tan}\left(\frac{x}{2} + \frac{\pi}{4}\right)$

(*xiii*) $\ln \operatorname{Tan}^{-1} x$

(*xiv*) $\ln \operatorname{Sin}^{-1} x^2$

(*xv*) 2^x

(*xvi*) x^x

(*xvii*) $(\operatorname{Tan} x)^{\tan x}$

(*xviii*) $x^{\ln x}$

(*xix*) $x^x + x^{\sin x}$

(*xx*) $x^{\sin x} + (\operatorname{Sin} x)^x$

(*xxi*) $(\ln x)^{\ln x}$

(*xxii*) $\frac{(x^4+1)^{\frac{3}{4}}}{(x^3+1)^{\frac{2}{3}}}$

(*xxiii*) $\frac{(x^2-1)^{\frac{1}{2}}(x+2)^3}{(x^3+2)^{\frac{1}{3}}}$

(*xxiv*) $\sqrt{\frac{a+x}{a-x}} \cdot \sqrt{\frac{b+x}{b-x}}$.

ANSWERS

Exercise 4.5.

2. (*i*) $\frac{1}{x\, ln_e{}^{10}} = \frac{1}{x} \log_{10} e = \frac{0.4343}{x}$

(*ii*) $1 + ln\, x$

(*iii*) $ln\, x$

(*iv*) $\frac{(1 - ln\, x)}{x^2}$

(*v*) $\frac{ln\, x - 1}{(in\, x)^2}$

(*vi*) $\left(\frac{1}{x} + ln\, x\right) e^x$

(*vii*) 1

(*viii*) $\frac{1}{x\, ln\, x}$

(*ix*) $\text{Cot}\, x$

(*x*) $\frac{\left(\text{Tan} \frac{1}{x}\right)}{x^2}$

(*xi*) $\text{Sec}\, x$

(*xii*) $\text{Sec}\, x$

(*xiii*) $\frac{1}{(1+x^2)\, \text{Tan}^{-1} x}$.

(*xiv*) $\frac{2x}{\text{Sin}^{-1} x^2 \sqrt{1-x^4}}$

(*xv*) $2^x\, ln\, 2$

(*xvi*) $x^x (1 + ln\, x)$

(*xvii*) $(\text{Tan}\, x)^{\tan x}\, \text{Sec}^2 x\, (1 + ln\, \text{Tan}\, x)^x$

(*xviii*) $x^{ln\, x} \cdot \frac{2}{x} ln\, x$

(*xix*) $x^x (1 + lnx) + x^{\text{Sin}\, x} \left(\frac{\text{Sin}\, x}{x} + \text{Cos} x ln\, x\right)$

(xx) $x^{\operatorname{Sin} x}\left(\operatorname{Cos} x \ln x+\frac{\operatorname{Sin} x}{x}\right)+(\operatorname{Sin} x)^{\prime}$
$(\ln \operatorname{Sin} x+x \operatorname{Cot} x)$

(xxi) $(\ln x)^{\ln x}\left(\frac{1+\ln \ln x}{x}\right)$

(xxii) $\frac{x^6+3x^3-2x^2}{(x^4+1)^{\frac{1}{4}}(x^3+1)^{\frac{5}{3}}}$

(xxiii) $\frac{(3x^5-2x^3+10x^2+4x-6)(x+2)^2}{(x^2-1)^{\frac{1}{2}}(x^3+2)^{\frac{4}{3}}}$

(xxiv) $\frac{(a+b)(ab-x^2)}{(a^2-x^2)(b^2-x^2)}\sqrt{\frac{(a+x)(b+x)}{(a-x)(b-x)}}$

Miscellaneous Exercise 2

1. Differentiate ab-initio the following functions :

(i) $\operatorname{Sec} \sqrt{x}$ (ii) $\operatorname{Sin} x^3$

(iii) $(\operatorname{Sin}^{-1} x)^2$ (iv) $\operatorname{Cos}^{-1} \sqrt{x}$

(v) $\operatorname{Cot}^{1} x^2$ (vi) $\operatorname{Cos}^3 x$

(vii) $e^{\operatorname{Sin} x}$ (viii) 2^{x^2}

(ix) e^{x^3} (x) $a^{\operatorname{Tan} x}$

2. Differentiate the following w.r.t. x :

(i) $\operatorname{Sec} x\sqrt{\operatorname{Sec} 2x}$

(ii) $\operatorname{Cosec}\sqrt{\operatorname{Cos} 2x}$

(iii) $(ax+b)^2\sqrt{1+\operatorname{Sin} 2x}$

(iv) $\operatorname{Sin}^3 (1+3x^2)^3$

(v) $[\operatorname{Cot}^2 (ax+b)^3]$

(vi) $\operatorname{Sin}^2 x \operatorname{Cos}^3 x$

3. Find the derivative of the follbwing w.r.t. x :

(*i*) $\text{Tan}^{-1}(\text{Sec}\, x+\text{Tan}\, x)$

(*ii*) $\text{Sin}^{-1} \dfrac{a+b\,\text{Cos}\, x}{b+a\,\text{Cos}\, x}$

(*iii*) $\text{Tan}^{-1} \sqrt{\dfrac{1-\text{Cos}\, x}{1+\text{Cos}\, x}}$

(*iv*) $\text{Cos}^{-1}\left(\dfrac{a}{b} \sqrt{\dfrac{x^2+b^2}{x^2+a^2}}\right)$

(*v*) $(2x+3)^2\, \text{Cot}^{-1} \dfrac{x}{1+x}$

(*vi*) $\text{Cosec}\,(4x-1)\, \text{Tan}^{-1} \dfrac{x}{x^2+1}$

4. Differentiate the following w.r.t. x.

(i) $ln\sqrt{\dfrac{x\,(x+1)}{(x+2)(x+3)}}$

(*ii*) $e^x\, \text{Cos}\, \sqrt{x}$

(*iii*) $ln\left(x^3\, e^{\sin^{-1} x}\right)$

(*iv*) $\sqrt{a^{\text{Tan}\sqrt{x}}}$

(*v*) $ln\, \dfrac{1+\text{Sin}\, x}{1-\text{Sin}\, x}$

(*vi*) $x^m\,(m ln x-1)$

(*vii*) $\text{Cot}^{-1} x\; ln\; \text{Sec}\, x$

(*viii*) $e^{a\, ln\, \cot x}$

(*ix*) $(\text{Sin}\, x)^{\text{Tan}\, x}+(\text{Tan}\, x)^{\text{Sin}\, x}$

(*x*) $x^x+(\text{Sec}\, x)^{\text{Cosec}\, x}$

5. Find $\frac{dy}{dx}$, if

(i) $x = e^{\theta} \text{Cos } \theta, \quad y = e^{\theta} \text{Sin } \theta$

(ii) $x = ln \sqrt{\frac{t^2+1}{t^2-1}}, \; y = \sqrt{\frac{t^2-1}{t^2+1}}$

6. If $y = \text{Sec } x \text{ Tan } x + ln \text{ Tan}\left(\frac{\pi}{2} + \frac{x}{2}\right)$, show that $\frac{dy}{dx}$ $= 2 \text{ Sec}^3 x$.

7. If $x = \text{Cos } \theta + ln \text{ Tan } \frac{\theta}{2}$ and $y = \text{Sin } \theta$, show that $\frac{dy}{dx} = \text{Tan } \theta$.

8. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{ln\ x}{(1+ln\ x)^2}$.

9. If $y = ln\,(x + \sqrt{x^2+a^2})$, prove that $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+a^2}}$.

10. If $y = e^{ax} \text{ Sin } (bx+c)$, show that

$$\frac{dy}{dx} = \sqrt{a^2+b^2}\, e^{ax} \text{ Sin } (bx + c + \tan^{-1} \frac{b}{a}).$$

ANSWERS

1. (i) $\frac{1}{2\sqrt{x}} \text{ Sec}\sqrt{x}. \text{ Tan } \sqrt{x}$

(ii) $3x^2 \text{ Cos } x^3$

(iii) $\frac{2 \text{ Sin}^{-1} x}{\sqrt{1-x^2}}$

(iv) $\frac{-1}{2\sqrt{x}\sqrt{1-x^2}}$

(*v*) $\frac{-2x}{1-x^4}$

(*vi*) $-3\ \text{Cos}^2x\ \text{Sin}\ x$

(*vii*) $e^{\text{Sin}\ x}\text{Cos}\ x$

(*viii*) $2x.\ 2^{x^2}\ ln\ 2$

(*ix*) $3x^2.\ e^{x^3}$

(*x*) $\text{Sec}^2x.\ a^{\text{Tan}\ x}\ lna$

2. (*i*) $\text{Sec}\ x\frac{\text{Sec}\ 2x\ \text{Tan}\ 2x}{\sqrt{\text{Sec}\ 2x}}+\text{Sec}\ x\ \text{Tan}\ x\sqrt{\text{Sec}\ 2x}$

(*ii*) $\frac{\text{Sin}\ 2x}{\sqrt{\text{Cos}\ 2x}}\text{Cosec}\sqrt{\text{Cos}\ 2x}\ \text{Cot}\sqrt{\text{Cos}\ 2x}$

(*iii*) $ax+b)^2\frac{\text{Cos}\ 2x}{\sqrt{1+\text{Sin}\ 2x}}+\sqrt{1+\text{Sin}\ 2x}.\ 2a\ (ax+b).$

(*iv*) $54x(1+3x^2)^2\ \text{Sin}^2\ (1+3x^2)^3\ \text{Cos}\ (1+3x^2)^3$

(*v*) $-6a\ (ax+b)^2\ \text{Cot}\ (ax+b)^3\ \text{Cosec}^2\ (ax+b)^3$

(*vi*) $\text{Cos}^2x\ \text{Sin}\ x\ (2\ \text{Cos}^2x-3\ \text{Sin}^2x)$

3. (*i*) $\frac{1}{2}$

(*ii*) $\frac{-\sqrt{b^2-a^2}}{b+a\ \text{Cos}\ x}$

(*iii*) $\frac{1}{2}$

(*iv*) $\frac{a\sqrt{b^2-a^2}}{(x^2+a^2)\sqrt{x^2+b^2}}$

(*v*) $4\ (2x+3)\ \text{Cot}^{-1}\frac{x}{1+x}-\frac{(2x+3)^2}{1+2x^2+2x}$

(*vi*) $\frac{1-x^2}{x^4+2x^2+1}\ \text{Cosec}\ (4x-1)-4\ \text{Cosec}(4x-1)$

$.\text{Cot}\ (4x-1)\ \text{Tan}^{-1}\frac{x}{x^2+1}$

4. (i) $\frac{1}{2}\left[\frac{1}{x}+\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}\right]$

(ii) $e^x\left[\frac{-\mathrm{Sin}\sqrt{x}}{2\sqrt{x}}+\mathrm{Cos}\sqrt{x}\right]$

(iii) $\frac{3}{x}+\frac{1}{\sqrt{1-x^2}}$

(iv) $\frac{1}{4\sqrt{x}}(\ln a)(a^{\mathrm{Tan}\sqrt{x}})^{\frac{1}{2}}\,\mathrm{Sec}^2\sqrt{x}$

(v) $2\,\mathrm{Sec}\,x$

(vi) $m^2x^{m-1}\ln x$

(vii) $\frac{-1}{1+x^2}\ln\mathrm{Sec}\,x+\mathrm{Tan}\,x\,\mathrm{Cot}^{-1}x$

(viii) $-a\,\mathrm{Sec}\,x\,\mathrm{Cosec}\,x\,e^{a\ln\mathrm{Cot}\,x}$

(ix) $(\mathrm{Sin}\,x)^{\mathrm{Tan}\,x}(1+\mathrm{Sec}^2x\ln\mathrm{Sin}\,x)+(\mathrm{Tan}\,x)^{\mathrm{Sin}\,x}$
$.(\mathrm{Sec}\,x+\mathrm{Cos}\,x\ln\mathrm{Tan}\,x)$

(x) $x^x(1+\ln x)+(\mathrm{Sec}\,x)^{\mathrm{Cosec}\,x}$
$.(\mathrm{Sec}\,x-\mathrm{Cosec}\,x\;\mathrm{Cot}\,x\ln\mathrm{Sec}\,x)$

5. (i) $\frac{\mathrm{Cos}\,\theta+\mathrm{Sin}\,\theta}{\mathrm{Cos}\,\theta-\mathrm{Sin}\,\theta}$

(ii) $-\sqrt{\frac{t^2-1}{t^2+1}}$

CHAPTER V

METHODS OF INTEGRATION

5.1. *The differential of a function.* For the function $y = f(x)$, we define

(*i*) dx, called differential of x, by the relation $dx = \triangle x$. (1)

(*ii*) dy, called the differential of y, by the relation $dy = f'(x)\, dx$. (2)

The differential of the independent variable is by definition equal to the increment of the variable, but the differential of the dependent variable is not equal to the increment of that variable.

It follows from (2) that differentials dx and dy have the following properties :—

(*a*) If $dx = 0$ then $dy = 0$, and

(*b*) If $dx \neq 0$ then

$$\begin{aligned}\frac{dy}{dx} &= \frac{\text{the differential of } y}{\text{the differential of } x}\\ &= \frac{f'(x)dx}{dx}\\ &= f'(x).\end{aligned}$$

Example 1. If $y = 8x$, then $f'(x) = 8$ and hence $dy = 8dx$.

Example 2. If $y = \text{Cos } x$, then $dy = -\text{Sin } x\, dx$.

5.2. *Standard differential formulae.* The following formulae for differentials can be derived from the corresponding formulae for derivatives.

(*i*) If $y = c$, c a constant, then $dy = 0$ or $dc = 0$.

(*ii*) If $y=x^n$, n being a real number,
$dy=nx^{n-1}dx$, or $d(x^n)=nx^{n-1}dx$.

(*iii*) If $y=u+v$, then $dy=du+dv$, or
$d(u+v)=du+dv$.

(*iv*) If $y=uv$, then $dy=udv+vdu$, or
$d(uv)=udv+vdu$.

(*v*) If $y=\frac{u}{v}$, then $dy=\frac{vdu-udv}{v^2}$

or $d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$

(*vi*) $d\,(\text{Sin } x)=\text{Cos } x\, dx$.

(*vii*) $d\,(\text{Cos } x)=-\text{Sin } x\, dx$.

(*viii*) $d\,(\text{Tan } x)=\text{Sec}^2x\, dx$.

(*ix*) $d\,(\text{Cot } x)=-\text{Cosec}^2x dx$.

(*x*) $d\,(\text{Sec } x)=\text{Sec } x \text{ Tan } x\, dx$.

(*xi*) $d\,(\text{Cosec } x)=-\text{Cosec } x \text{ Cot } x dx$.

(*xii*) $d\,(\text{Sin}^{-1}x)=\frac{1}{\sqrt{1-x^2}}\,dx$.

(*xiii*) $d\,(\text{Cos}^{-1}x)=-\frac{1}{\sqrt{1-x^2}}\,dx$.

(*xiv*) $d\,(\text{Tan}^{-1}x)=\frac{1}{1+x^2}\,dx$.

(*xv*) $d\,(\text{Cot}^{-1}x)=-\frac{1}{1+x^2}\,dx$.

(*xvi*) $d\,(\text{Sec}^{-1}x)=\frac{1}{x\sqrt{x^2-1}}\,dx$.

(*xvii*) $d\,(\text{Cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}\,dx$.

(*xviii*) $d(\ln x) = \dfrac{1}{x}\,dx.$

(*xix*) $d(e^x) = e^x dx.$

5.3. *Approximation by Differentials.* If $dx \cong \Delta x$ is small when compared with x, dy gives a good approximation of Δy.

Example 1. Let $y = x^2 + x + 1$ and let x change from $x = 2$ to $x = 2.01$. The actual change in y is

$$\begin{aligned}\Delta y &= \{(2.01)^2 + 2.01 + 1\} - \{2^2 + 2 + 1\}\\ &= .0501.\end{aligned}$$

The approximate change in y is

$$\begin{aligned}dy &= f'(x)dx\\ &= (2x+1)dx\\ &= \{2(2)+1\}(.01) \quad (\text{taking } x = 2 \text{ and } dx = .01)\\ &= .05.\end{aligned}$$

Example 2. Use differentials to find.

(*i*) $\sqrt[3]{124}.$ (*ii*) Sin 60° 1′

Solution. (*i*) Let $y = x^{\frac{1}{3}}$.

Then $dy = \dfrac{1}{3x^{\frac{2}{3}}}dx.$

Take $x = 125 = 5^3$ and $dx = -1$.

Then $dy = \dfrac{1}{3(125)^{\frac{2}{3}}}(-1) = -\dfrac{1}{75} = -0.0$

Hence $\sqrt[3]{124} = y + dy = 5 - 0.0133 \approx 4.9867.$

(*ii*) Let $y = \text{Sin } x$

Then $dy = \text{Cos } x \, dx$

Take $x = 60°$ and $dx = 1' = 0.0003$ rad.

Then $dy = \text{Cos } 60°(0.0003)$

$= \frac{1}{2}(0.0003)$

$= 0.00015.$

Hence $\text{Sin } 60° 1' = y + dy = \text{Sin } 60° + 0.00015$

$= 0.86603 + 0.00015$

$= 0.86618.$

Example 3. Use differentials to obtain $\frac{dy}{dx}$, if $xy + x - 2y = 5$.

Solution. $xy + x - 2y = 5$

$\Rightarrow d(xy) + d(x) - d(2y) - d(5)$

$\Rightarrow x(dy) + ydx + dx - 2dy = 0$

$\Rightarrow (x-2)dy + (y+1)dx = 0.$

Hence $\frac{d}{dx} = -\frac{y+1}{x-2}.$

Exercise 5.1

1. Find dy for each of the following :

(i) $y\ (5-x)^3$.

(ii) e^{4x^2}

(iii) $y = \frac{\text{Sin } x}{x}$.

(iv) $y = \text{Cos } bx^2$.

(v) $y = \text{arc Cos } 2x$.

(vi) $y = ln \text{ Tan } x$.

2. Using differentials, differentiate the following :

(i) $2xy^3 + 3x^2y = 1$.

(ii) $x^3y^2 - 2x^2y + 3xy^2 - 8xy = 6$.

(*iii*) arc Tan $\frac{y}{x} = ln(x^2 + y^2)$.

(*iv*) $xy = \text{Sin } (x - y)$.

(*v*) $x^2 lny + y^2 lnx = 2$.

3. Use differentials to approximate :

(*i*) $\sqrt[4]{17}$. (*ii*) $\sqrt{1020}$.

(*ii*) Cos 59°. (*iv*) Tan 44 .

4. Use differentials to approximate the change in :

(*i*) x^3 as x changes from 5 to 5.01.

(*iii*) $1/x$ as x changes from 1 to 0.98.

5. A circular plate expands under the influence of heat so that its radius increases from 5 in. to o.06 in. Find the approximate increase in area.

6. A sphere of ice of radius 10 in. shrinks to radius 9.8 in. Approximate the decrease in (*i*) volume and (*ii*) surface area.

7. Find the approximate change in the volume of a cube of side x in. caused by increasing the sides by 1%.

8. Find the change in the total surface of a right circular cone when (*i*) the radius remains constant while the altitude changes by a small amount. (*ii*) the altitude remains constant while the radius changes by a small amount.

ANSWERS

1. (*i*) $-3(5 - x^2 dx$.

(*ii*) $8xe^{4x^2} dx$.

(*iii*) $\frac{x\text{Cos } x - \text{Sin } x}{x^2} dx$.

(*iv*) $-2bx \operatorname{Sin} bx^2 dx$.

(*v*) $\dfrac{-2}{\sqrt{1-4x^2}}\,dx$.

(*vi*) $\dfrac{2dx}{\operatorname{Sin} 2x}$

2. (*i*) $-\dfrac{-2y(y^2+3x)}{3x(2y^2+x)}$

(*ii*) $\dfrac{4xy-3x^2y^2-3y^2+8y}{6xy+2x^3y-2x^2-8x}$.

(*iii*) $\dfrac{2x+y}{x-2y}$.

(*iv*) $\dfrac{\operatorname{Cos}(x-y)-y}{\operatorname{Cos}(x-y)+x}$

(*v*) $-\dfrac{(2x^2 \ln y+y^2)y.}{(2y^2 \ln x+x^2)x}$.

3. (*i*) 2.03125. (*ii*) 31 9375.

(*iii*) .5150. (*iv*) 0.9651.

4. (*i*) 0.75. (*ii*) 0.02.

5. $0.6\pi = 1.88$ (in).^{2}

6. (*i*) 80π in.^{3} (*ii*) 16π in.^{2}

7. $0.03x^3$ in.^{3}

8. (*i*) $\dfrac{\pi r h\,dh}{\sqrt{r^2+h^2}}$

(*ii*) $\left\{\dfrac{h^2+2r^2}{\sqrt{r^2+h^2}} \pm 2r\right\} dr$.

5.4. *Standard formulae for Integration*

1. $\int (ax+b)^n d(ax+b) = \frac{(ax+b)^{n+1}}{n+1} + c,\ n \neq -1.$

2. $\int \frac{d(ax+b)}{(ax+b)} = ln(ax+b) + c,\ (ax+b) > 0.$

3. $\int a^{ax+b} d(ax+b) = \frac{a^{ax+b}}{lna} + c,\ a > 0,\ a \neq 1.$

4. $\int e^{ax+b} d(ax+b) = e^{ax+b} + c.$

5. $\int \text{Sin}\ (ax+b) d(ax+b) = -\text{Cos}\ (ax+b) + c.$

6. $\int \text{Cos}\ (ax+b) d(ax+b) = \text{Sin}\ (ax+b) + c.$

7. $\int \text{Tan}\ (ax+b) d(ax+b) = ln[\text{Sec}\ (ax+b)] + c.$

8. $\int \text{Cot}(ax+b) d(ax+b) = ln[\text{Sin}(ax+b)] + c.$

9. $\int \text{Sec}(ax+b) d(ax+b) = ln[\text{Sec}(ax+b) + \text{Tan}(ax+b)] + c.$

10. $\int \text{Cosec}(ax+b) d(ax+b) = ln[\text{Cosec}(ax+b) - \text{Cot}(ax+b)] + c.$

11. $\int \text{Sec}^2(ax+b) d(ax+b) = \text{Tan}(ax+b) + c.$

12. $\int \text{Cosec}^2(ax+b) d(ax+b) = -\text{Cot}(ax+b) + c.$

13. $\int \text{Sec}(ax+b) \text{Tan}(ax+b) d(ax+b) = \text{Sec}(ax+b) + c.$

14. $\int \text{Cosec}(ax+b) \text{Cot}(ax+b) d(ax+b) = -\text{Cosec}(ax+b) + c.$

15. $\int \frac{dx}{\sqrt{a^2-x^2}} = \text{arc Sin}\,\frac{x}{a} + c.$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\,\text{arc Tan}\,\frac{x}{a} + c$

17. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\,\text{arc Sec}\,\frac{x}{a} + c.$

18. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\, ln \left(\frac{x-a}{x+a}\right) + c.$

19. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\, ln \left(\frac{a+x}{a-x}\right) + c.$

20. $\int \frac{dx}{\sqrt{x^2+a^2}} = ln\,(x+\sqrt{x^2+a^2}) + c.$

21. $\int \frac{dx}{\sqrt{x^2-a^2}} = ln\,(x+\sqrt{x^2-a^2}) + c.$

22. $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\,\text{arc Sin}\,\frac{x}{a} + c.$

23. $\int \sqrt{x^2+a^2\,dx} = \frac{1}{2}x\sqrt{x^2+a^2}$
$+ \frac{1}{2}a^2\, ln\,(x+\sqrt{x^2+a^2}) + c$

24. $\int \sqrt{x^2-a^2\,dx} = \frac{1}{2}x\sqrt{x^2-a^2}$
$- \frac{1}{2}a^2\, ln\,(x+\sqrt{x^2-a^2}) + c$

These formulae can be established by showing the derivativ w.r.t.x, of the right hand side equal to the integrand Thus for example in the formula (22)

$$\frac{d}{dx}\left(\frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\,\text{arc Sin}\,\frac{x}{a} + c\right)$$

$$= \frac{1}{2}\sqrt{a^2-x^2} + \frac{x(-2x)}{4\sqrt{a^2-x^2}} + \frac{1}{2}a^2\frac{1/a}{\sqrt{1-\frac{x^2}{a^2}}} + 0$$

$$= \frac{1}{2}\sqrt{a^2-x^2} - \frac{1}{2}\frac{x^2}{\sqrt{a^2-x^2}} + \frac{a^2}{2\sqrt{a^2-x^2}}$$

$$=\tfrac{1}{2}\sqrt{a^2-x^2}+\frac{(a^2-x^2)}{2\sqrt{a^2-x^2}}=\sqrt{a^2-x^2}.$$

Theorem 5.4.1. $\int\frac{f'(x)}{f(x)}\,dx=ln\,f(x)+c.$

This follows at once, since

$$\frac{d}{dx}\left[\,ln\,f(x)+c\,\right]=\frac{f'(x)}{f(x)}.$$

Exercise 5.2.

Integrate

1. $\int\frac{dx}{x^2-1}$
2. $\int\frac{dx}{1-x^2}$
3. $\int\frac{dx}{x^2-4}$
4. $\int\frac{dx}{9-x^2}$
5. $\int\frac{dx}{\sqrt{x^2+1}}$
6. $\int\frac{dx}{\sqrt{x^2-1}}$
7. $\int\sqrt{25-x^2}\,dx$
8. $\int\sqrt{x^2-36}\,dx$

ANSWERS

1. $\frac{1}{2}\, ln\left(\frac{x-1}{x+1}\right) + c.$

2. $\frac{1}{2}\, ln\left(\frac{1+x}{1-x}\right) + c.$

3. $\frac{1}{4}\, ln\left(\frac{x-2}{x+2}\right) + c.$

4. $\frac{1}{6}\, ln\left(\frac{3+x}{3-x}\right) + c.$

5. $ln\,(x \tfrac{1}{2} \sqrt{x^2+1)} + c.$

6. $ln\,(x+\sqrt{x^2-1}\,) + c$

7. $\frac{1}{2}\, x\sqrt{25-x^2} + \frac{25}{2}\text{ arc Sin}\frac{x}{5} + c.$

8. $\frac{1}{2}\, x\sqrt{x^2-36} - 18 ln\left(x+\sqrt{x^2-36}\right) + c.$

5.5. *Finding Indefinite Integrals by Substitution.* Sometimes the change of the variable x in $\int f(x)\,dx$ by means of a substitution of the form $x=\varnothing(u)$ changes the integral to a standard form or to a form easier to deal with as illustrated in the following examples :

Example 1. Find $\int \frac{xdx}{\sqrt{4+x^2}}$.

Solution. Let $n=4+x^2$, then $du=2x\,dx$.

Hence $\int \frac{xdx}{\sqrt{4+x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-\frac{1}{2}}du = \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)+c$

$= u^{\frac{1}{2}} + c = \sqrt{4+x^2} + c.$

Example 2. Find $\int 3\,(2x+1)^5 dx$.

Solution. Let $u=2x+1$. Then $du=2dx$, $dx=\frac{1}{2}\,du$

and $\int 3\,(2x+1)^5 dx=\int 3u^5 \frac{1}{2}\,du=\frac{3}{2}\int u^5\,du$

$$=\frac{3}{2}\cdot\frac{u^6}{6}+c=\frac{1}{4}\,(2x+1)^6+c.$$

Example 3. Find $\int \text{Cos } 3x\,dx$.

Solution. Let $u=3x$, then $du=3dx$, $dx=\frac{1}{3}\,du$

and $\int \text{Cos } 3x\,dx=\frac{1}{3}\int \text{Cos } u\,du=\frac{1}{3}\,\text{Sin } u+c$

$$=\frac{1}{3}\,\text{Sin } 3x+c.$$

Example 4. Find $\int e^{\text{Tan } \theta}\,\text{Sec}^2\,\theta\,d\,\theta$.

Solution. Let $\text{Tan } \theta=t$. Then $\text{Sec}^2\,\theta d\theta=dt$

$$\therefore\ \int e^{\text{Tan } \theta}\,\text{Sec}^2\,\theta d\theta=\int e^t\,dt=e^t+e=e^{\text{Tan } \theta}+c.$$

Example 5 Find $\int \frac{xdx}{\sqrt{x^2+1}}$.

Solution. Let $u=x^2+1$.

Then $du=2x\,dx$, $xdx=\frac{1}{2}\,du$ and

$$\int \frac{xdx}{\sqrt{x^2+1}}=\frac{1}{2}\int u^{\frac{1}{2}}du=\frac{1}{2}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c=\sqrt{x^2+1}.$$

Example 6. Find $\int \text{Cos}^3\,6x\ \text{Sin } 6x\,dx$

Solution. Let $u=\text{Cos } 6x$. Then $du=-6\,\text{Sin } 6x\,dx$..

Thus $\int \text{Cos}^3\,6x\ \text{Sin } 6x\,dx=\int u^3.\ \left(-\frac{du}{6}\right)$.

$$=-\frac{1}{6}\int u^3\,du=-\frac{1}{6}\frac{u^4}{4}+c=-\frac{1}{24}\,\text{Cos}^4\,6x+c.$$

Example 7. Find (*i*) $\int \text{Cosec}\, dx$ (*u*) $\int \text{Sec}\, x dx$.

Solution (*i*) $\int \text{Cosec}\, x dx = \int \frac{dx}{\text{Sin}\, x} = \int \frac{dx}{2\, \text{Sin}\, \frac{x}{2} \text{Cos}\, \frac{x}{2}}$

$$= \int \frac{\frac{1}{2}\, \text{Sec}^2 \frac{x}{2}\, dx}{\text{Tan}\, \frac{x}{2}}$$

Let $\text{Tan} \frac{x}{2} = t$. Then $\frac{1}{2}\, \text{Sec}^2 \frac{x}{2}\, dx = dt$ and

$\int \text{Cosec}\, x\, dx = \int \frac{dt}{t} = ln t + c = ln\, \text{Tan} \frac{x}{2} + c.$

(*ii*) $\int$ **Sec** ***xdx*** **can be found similarly. The value of** $\int \text{Sec}\, x\, dx$ can also be deduced from that of $\int \text{Cosec}\, x\, dx$ as follows :

$$\int \text{Sec}\, x\, dx = \int \frac{dx}{\text{Cos}\, x} = \int \frac{dx}{\text{Sin}\left(\frac{\pi}{2} + x\right)} = \int \frac{d\left(\frac{\pi}{2} + x\right)}{\text{Sin}\left(\frac{\pi}{2} + x\right)}$$

Let $u = \frac{\pi}{2} + x$. Then

$$\int \text{Sec}\, x\, dx = \int \frac{du}{\text{Sin}\, u} = ln\, \text{Tan}\, \frac{u}{2} + c$$

$$= ln\ \text{Tan}\left(\frac{\pi}{4} + \frac{x}{2}\right) + c.$$

Exercise 5.3

1. *Evaluate* :

(*i*) $\int (x^3+2)^2 3x^2 dx$

(*ii*) $\int (x^3+2)^{\frac{1}{2}} x^2\, dx$

(*iii*) $\int \frac{8x^2\, dx}{(x^3+2)^3}$

(*iv*) $\int \frac{x^2\, dx}{\sqrt[4]{x^3+2}}$

2. Evaluate : $\int 3x\sqrt{1-2x^2}\, dx$

3. Evaluate : $\int \frac{(x+3)\, dx}{(x^2+6x)^{\frac{1}{3}}}$

4. *Evaluate*

(*i*) $\int \sqrt[3]{1-x^2}\, x\, dx$

(*ii*) $\int \sqrt{x^2-2x^4}\, dx$

5. *Evaluate* :

(*i*) $\int \frac{dx}{x+2}$

(*ii*) $\int \frac{dx}{2x+3}$

(*iii*) $\int \frac{x\, dx}{x^2-1}$

(*iv*) $\int \frac{x^2\, dx}{1-2x^3}$

(*v*) $\int \frac{x+2}{x+1}\, dx.$

6. ***Evaluate*** ·

(*i*) $\int e^{-x}\, dx$

(*ii*) $\int e^{3x}\, dx$

(*iii*) $\int a^{2x}\, dx$

(*iv*) $\int \frac{e^{x^{-1}}}{x^2} dx$

(*v*) $\int (e^x+1)^3 e^x\, dx$

(*vi*) $\int \frac{dx}{e^x+1}$

7. ***Evaluate*** :

(*i*) $\int \text{Sin}\, \frac{1}{2} x\, dx$

(*ii*) $\int \text{Cos}\, 3x\, dx$

(*iii*) $\int \text{Sin}^2 x\, \text{Cos}\, x\, dx$

(*iv*) $\int \text{Tan}\, x\, dx$

(*v*) $\int \text{Tan}\, 2x\, dx$

(*vi*) $\int x\, \text{Cot}\, x^2\, dx$

(*vii*) $\int \text{Sec}\sqrt{x}\, \frac{dx}{\sqrt{x}}$

(*viii*) $\int \text{Sec}^2 2ax\, dx$

(*ix*) $\int \frac{\text{Sin } x + \text{Cos } x}{\text{Cos } x} dx$

(*x*) $\int \frac{\text{Sin } y\, dy}{\text{Cos}^2 y}$

(*xi*) $\int (1 + \text{Tan } x)^2\, dx$

(*xii*) $\int e^x \text{Cos } e^x dx$

(*xiii*) $\int e^{3 \cos 2x} 2 \text{ Sin } 2x\, dx$

(*xiv*) $\int (\text{Tan } 2x + \text{Sec } 2x)^2\, dx$

(*xv*) $\int \text{Sec } 4x - 1)^2\, dx$

(*xvi*) $\int \frac{\text{Sec } x \text{ Tan } x\, dx}{a + b \text{ Sec } x}$

(*xvii*) $\int \frac{dx}{\text{Cosec } 2x - \text{Cot } 2x}$

8. *Evaluate* :

(*i*) $\int \frac{dx}{\sqrt{25 - 16x^2}}$

(*ii*) $\int \frac{dx}{4x^2 + 9}$

(*iii*) $\int \frac{dx}{x\sqrt{4x^2 - 9}}$

(*iv*) $\int \frac{x^2 dx}{\sqrt{1 - x^6}}$

(*v*) $\int \frac{x\, dx}{x^4 + 3}$

(vi) $\int \frac{dx}{x\sqrt{x^4-1}}$

(vii) $\int \frac{dx}{\sqrt{4-(x+2)^2}}$

(viii) $\int \frac{dx}{e^x+e^{-x}}$

(ix) $\int \frac{\operatorname{Sec} x \operatorname{Tan} x \, dx}{9+4 \operatorname{Sec}^2 x}$

(x) $\int \frac{(x+3)\,dx}{\sqrt{1-x^2}}$

(xi) $\int \frac{(2x-7)\,dx}{x^2+9}$

(xii) $\int \frac{dt}{1+(3t-4)^2}$

9. *Evaluate* :

(i) $\int \frac{dx}{\sqrt{4x^2+9}}$

(ii) $\int \frac{dz}{\sqrt{9z^2-25}}$

(iii) $\int \frac{dx}{9x^2-16}$

(iv) $\int \frac{dy}{25-16y^2}$

10. *Integrate* :

(i) $\sqrt{3-4x^2}$

(ii) $\sqrt{3x^2-5}$

(iii) $\sqrt{16-9x^2}$

(*iv*) $\sqrt{4x^2+9}$

11. By using trigonometric identities or otherwise evaluate the following :—

(*i*) $\int \text{Sin}^2 x \, dx$

(*ii*) $\int \text{Cos}^2 3x \, dx$

(*iii*) $\int \text{Sin}^3 x \, dx$

(*iv*) $\int \text{Cos}^5 x \, dx$

(*v*) $\int \text{Sin}^2 x \, \text{Cos}^3 x \, dx$

(*vi*) $\int \text{Cos}^4 2x \, \text{Sin}^3 2x \, dx$

(*vii*) $\int \text{Sin}^3 3x \, \text{Cos}^5 3x \, dx$

(*viii*) $\int \text{Cos}^3\left(\frac{x}{3}\right) dx$

(*ix*) $\int \text{Sin}^4 x \, dx$

(*x*) $\int \text{Sin}^2 x \, \text{Cos}^2 x \, dx$

(*xi*) $\int \text{Sin}^4 3x \, \text{Cos}^3 3x \, dx$

(*xii*) $\int \text{Sin } 3x \, \text{Sin } 2x \, dx$

(*xiii*) $\int \text{Sin } 3x \, \text{Cos } 5x \, dx$

(*xiv*) $\int \text{Cos}\, 4x \,\text{Cos}\, 2x \, dx$

(*xv*) $\int \sqrt{1 - \text{Cos}\ x}\; dx$:

(*xvi*) $\int (1 + \text{Cos}\, 3x)^{\frac{3}{2}} \, dx$

(*xvii*) $\int \frac{dx}{\sqrt{1 - \text{Sin}\, 2x}}$

12. By using trigonometric identities or otherwise, evaluate the following :

(*i*) $\int \text{Tan}^4 x \, dx$

(*ii*) $\int \text{Tan}^5 x \, dx$

(*iii*) $\int \text{Sec}^4 2x \, dx$

(*iv*) $\int \text{Tan}^3 3x \,\text{Sec}^4 3x \, dx$

(*v*) $\int \text{Tan}^3 2x \,\text{Sec}^3 2x \, dx$

(*vi*) $\int \text{Cot}^3 2x \, dx$

(*vii*) $\int \text{Cot}^4 3x \, dx$

(*viii*) $\int \text{Cosec}^6 x \, dx$

(*ix*) $\int \text{Cot}\, 3x \,\text{Cosec}^4 3x \, dx$

ANSWERS

1. (*i*) $\frac{1}{3}(x^3+2)^3+c$

(*ii*) $\frac{2}{9}(x^3+2)^{\frac{3}{2}}+c$

(*iii*) $\dfrac{-4}{3(x^3+2)^2}+c$

(*iv*) $\frac{4}{9}(x^3+2)^{\frac{3}{4}}+c$

2. $-\frac{1}{2}(1-2x^2)^{\frac{3}{2}}+c$

3. $\frac{3}{4}(x^2+6x)^{\frac{2}{3}}+c$

4. (*i*) $-\frac{3}{8}(1-x^2)^{\frac{4}{3}}+c$

(*ii*) $-\frac{1}{6}(1-2x^2)^{\frac{3}{2}}+c$

5. (*i*) $ln\,(x+2)+c$

(*ii*) $\frac{1}{2}\,ln\,(2x+3)+c$

(*iii*) $ln\,c\sqrt{x^2-1}$

(*iv*) $ln\,\dfrac{c}{\sqrt[6]{(1-2x^3)}}$

(*v*) $x+ln\,(x+1)+c$

6. (*i*) $-e^{-x}+c$

(*ii*) $\dfrac{e^{3x}}{3}+c$

(*iii*) $\frac{1}{2}\left(\dfrac{a^{2x}}{lna}\right)+c$

(*iv*) $-e^{\frac{1}{x}}+c$

(*v*) $\dfrac{(e^x+1)^4}{4}+c$

(*vi*) $x-ln\,(e^x+1)+c$

7. (*i*) $-2\,\text{Cos}\,\frac{1}{2}x + c$

(*ii*) $\frac{1}{3}\,\text{Sin}\,3x + c$

(*iii*) $\dfrac{\text{Sin}^3 x}{3} + c$

(*iv*) $ln\,(\text{Sec}\,x) + c$

(*v*) $\frac{1}{2}\,ln\,(\text{Sec}\,2x) + c$

(*vi*) $\frac{1}{2}ln\,(\text{Sin}\,x^2) + c$

(*vii*) $2\,ln\,(\text{Sec}\sqrt{x} + \text{Tan}\sqrt{x}) + c$

(*viii*) $\dfrac{\text{Tan}\,2ax}{2a} + c$

(*ix*) $ln\,(\text{Sec}\,x) + x + c$

(*x*) $\text{Sec}\,y + c$

(*xi*) $\text{Tan}\,x + 2\,ln\,(\text{Sec}\,x) + c$

(*xii*) $\text{Sin}\,e^x + c$

(*xiii*) $\dfrac{-e^{3\cos 3x}}{3} + c$

(*xiv*) $\text{Tan}\,2x + \text{Sec}\,2x - x + c$

(*xv*) $\frac{1}{4}\,\text{Tan}\,{}^4x - \frac{1}{2}\,ln\,(\text{Sec}\,4x + \text{Tan}\,4x) + x + c$

(*xvi*) $\dfrac{1}{b}\,ln\,(a + b\text{Sec}\,x) + c$

(*xvii*) $ln\,(\text{Sin}\,x) + c.$

8. (*i*) $\frac{1}{4}$ arc $\text{Sin}\,\dfrac{4x}{5} + c$

(*ii*) $\frac{1}{6}$ arc $\text{Tan}\,\dfrac{2x}{3} + c$

(*iii*) $\frac{1}{3}$ arc $\text{Sec}\,\dfrac{2x}{3} + c$

(*iv*) $\frac{1}{3}$ arc $\text{Sin}\,x^3 + c$

(*v*) $\dfrac{\sqrt{3}}{6}$ arc $\text{Tan}\,\dfrac{x^2\sqrt{3}}{3} + c$

(*vi*) $\frac{1}{2}$ arc $\text{Sec}\,x^2 + c$

(*vii*) $\text{arc Sin}\dfrac{x+2}{2}+c$

(*viii*) $\text{arc Tan } e^x+c$

(*ix*) $\frac{1}{6}\text{ arc Tan }\dfrac{2\text{ Sec }x}{3}+c$

(*x*) $-\sqrt{1-x^2}+3\text{ arc Sin }x+c$

(*xi*) $ln\,(x^2+9)-\frac{7}{3}\text{ arc Tan}\dfrac{x}{3}+c$

(*xii*) $\frac{1}{3}\text{ Tan}^{-1}\,(3t-4)+c$

9. (*i*) $\frac{1}{2}\,ln\,(2x+\sqrt{4x^2+9})+c$

(*ii*) $\frac{1}{3}\,ln\,(3z+\sqrt{9z^2-25})+c$

(*iii*) $\dfrac{1}{24}\,ln\left(\dfrac{3x-4}{3x+4}\right)+c$

(*iv*) $\dfrac{1}{40}\,ln\left(\dfrac{5+4y}{5-4y}\right)+c$

10. (*i*) $\frac{1}{2}\,x\sqrt{3-4x^2}+\frac{3}{4}\text{ arc Sin}\dfrac{2x}{\sqrt{3}}+c$

(*ii*) $\frac{1}{2}\,x\sqrt{3x^2-5}-\dfrac{5\sqrt{3}}{6}\,ln\,(\sqrt{3}x+\sqrt{3x^2-5})+c$

(*iii*) $\frac{1}{2}\,x\sqrt{16-9x^2}+\frac{8}{3}\text{ arc Sin}\dfrac{3x}{4}+c$

(*iv*) $\frac{1}{2}\,x\sqrt{4x^2+9}+\frac{9}{4}\,ln\,(2x+\sqrt{4x^2+9})+c$

11. (*i*) $\frac{1}{2}\,x-\frac{1}{4}\text{ Sin }2x+c$

(*ii*) $\frac{1}{2}\,x+\frac{1}{12}\text{ Sin }6x+c$

(*iii*) $-\text{Cos }x+\frac{1}{3}\text{ Cos}^3x+c$

(*iv*) $\text{Sin }x-\frac{2}{3}\text{ Sin}^3\,x+\frac{1}{5}\text{ Sin}^5x+c$

(*vi*) $-\frac{1}{10}\text{ Cos}^5\,2x+\frac{1}{14}\text{ Cos}^7\,2x+c$

(*vii*) $-\frac{1}{18}\text{ Cos}^6\,3x+\frac{1}{24}\text{ Cos}^8\,3x+c$

(*viii*) $3\text{ Sin }\dfrac{x}{3}-\text{Sin}^3\dfrac{x}{3}+c$

(*ix*) $\frac{3}{8}x-\frac{1}{4}\operatorname{Sin}2x+\frac{1}{32}\operatorname{Sin}4x+c$

(*x*) $\frac{1}{8}x-\frac{1}{32}\operatorname{Sin}4x+c$

(*xi*) $\frac{1}{15}\operatorname{Sin}^5 3x-\frac{1}{21}\operatorname{Sin}^7 3x+c$

(*xii*) $\frac{1}{2}\operatorname{Sin}x-\frac{1}{10}\operatorname{Sin}5x+c$

(*xiii*) $\frac{1}{4}\operatorname{Cos}2x-\frac{1}{16}\operatorname{Cos}8x+c$

(*xiv*) $\frac{1}{4}\operatorname{Sin}2x+\frac{1}{12}\operatorname{Sin}6x+c$

(*xv*) $-2\sqrt{2}\operatorname{Cos}\frac{1}{2}x+c$

(*xvi*) $2\sqrt{2}\left(\frac{2}{3}\operatorname{Sin}\frac{3}{2}x-\frac{2}{9}\operatorname{Sin}^3\frac{3}{2}x\right)+c$

(*xvii*) $-\frac{\sqrt{2}}{2}\,ln\left[\operatorname{Cosec}\left(\frac{\pi}{4}-x\right)-\operatorname{Cot}\left(\frac{\pi}{4}-x\right)\right]+c$

12. (*i*) $\frac{1}{3}\operatorname{Tan}^3 x-\operatorname{Tan}x+x+c$

(*ii*) $\frac{1}{4}\operatorname{Tan}^4 x-\frac{1}{2}\operatorname{Tan}^2 x+ln\,(\operatorname{Sec}x)+c$

(*iii*) $\frac{1}{2}\tan 2x+\frac{1}{6}\operatorname{Tan}^3 2x+c$

(*iv*) $\frac{1}{12}\operatorname{Tan}^4 3x+\frac{1}{18}\operatorname{Tan}^6 3x+c$

(*v*) $\frac{1}{10}\operatorname{Sec}^5 2x-\frac{1}{6}\operatorname{Sec}^3 2x+c$

(*vi*) $-\frac{1}{4}\operatorname{Cot}^2 2x+\frac{1}{2}\,ln\,(\operatorname{Cosec}2x)+c$

(*vii*) $-\frac{1}{9}\operatorname{Cot}^3 3x+\frac{1}{3}\operatorname{Cot}3x+x+c$

(*viii*) $-\operatorname{Cot}x-\frac{2}{3}\operatorname{Cot}^3 x-\frac{1}{5}\operatorname{Cot}^5 x+c$

(*ix*) $-\frac{1}{6}\operatorname{Cot}^2 3x-\frac{1}{12}\operatorname{Cot}^4 3x+c$

5.6. *Substitution in Definite Integrals.*

We illustrate the use of substitution to comptute definite integrals.

Example 1. Evaluate $\int_1^2 (x+1)\sqrt[3]{x^2+2x+2}\,dx$.

Solution. Let $u=x^2+2x+2$- Then $du=2\,(x+1)\,dx$, $(x+1)\,dx=\frac{1}{2}\,du$. when $x=1$, $u=5$ and when $x=2$, $u=10$. Now

$$\int_1^2 (x+1)\sqrt[3]{x^2+2x+2}\,dx=\frac{1}{2}\int_5^{10} u^{\frac{1}{3}}\,du=\frac{1}{2}\left[\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\right]_5^{10}$$

$$=\frac{3}{8}\left(10^{\frac{4}{3}}-5^{\frac{4}{3}}\right).$$

Example 2. Evaluate $\int_{-2}^{-1} x\sqrt{2x^2+1}\,dx$.

Solution : Let $u = (2x^2+1)$.– Then $du = 4x\,dx$

$x\,dx = \frac{1}{4}du$.

When $x = -1$, $u = 3$ and when $x = -2$, $u = 9$ Now

$$\int_{-2}^{-1} x\sqrt{2x^2+1}\,dx = \frac{1}{4}\int_{9}^{3} u^{\frac{1}{2}}\,du = \left[\frac{1}{6}u^{\frac{3}{2}}\right]_9^3$$

$$= \frac{1}{6}\left[(3)^{\frac{3}{2}} - (9)^{\frac{3}{2}}\right]$$

Exercise 5.4

Evaluate the following integrals :—

1. $\int_0^3 \sqrt[3]{(3t-1)^2}\,dt$
2. $\int_{-2}^{1} \sqrt{2-x\,dx}$
3. $\int_{-3}^{-1} \frac{dx}{(x-1)^2}$
4. $\int_{-2}^{1} x\sqrt{x^2+1\,dx}$
5. $\int_0^1 \frac{(2x+3)\,dx}{\sqrt{x^2+3x+1}}$
6. $\int_1^2 (2x+1)\sqrt{x^2+x+1}\,dx$

7. Evaluate :

$$(i) \int_{\frac{-\pi}{3}}^{\frac{\pi}{6}} \mathrm{Sin}^2 x \,\mathrm{Cos}\, x \, dx$$

$$(ii) \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \mathrm{Cot}^2 2x \,\mathrm{Cosec}^2 2x dx$$

$$(iii) \int_{0}^{\pi} \mathrm{Cos}^2 x \, dx$$

$$(iv) \int_{0}^{\frac{\pi}{2}} \mathrm{Cos}^4 x \, dx$$

$$(v) \int_{0}^{\frac{\pi}{2}} \mathrm{Sin}^2 x \cdot \mathrm{Cos}^2 x \, dx$$

$$(vi) \int_{0}^{\frac{\pi}{3}} \mathrm{Tan}^3 x \,\mathrm{Sec}\, x \, dx$$

$$(vii) \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \operatorname{Tan}^4 x \operatorname{Sec}^2 x \, dx$$

ANSWERS

(1) $\frac{33}{5}$ (2) $\frac{14}{3}$ (3) $\frac{1}{4}$

(4) $\frac{1}{3}(2\sqrt{2}-5\sqrt{5})$ (5) $2(\sqrt{41}\pm 1)$

(6) $\frac{2}{3}[7^{\frac{3}{2}}-3^{\frac{3}{2}}]$

(7) (*i*) $\frac{1}{24}\left(1+3\sqrt{3}\right)$

(*ii*) $\frac{\sqrt{3}}{27}$ (*iii*) $\frac{\pi}{2}$

(*iv*) $\frac{3\pi}{16}$ (*v*) $\frac{\pi}{16}$

(*vi*) $\frac{4}{3}$ (*vii*) $\frac{2}{5}$

5.7. *Trigonometric Substitution.*

We now consider indefinite integrals of functions involving one of the forms

$$\sqrt{a^2-x^2},\ \sqrt{a^2+x^2} \text{ or } \sqrt{x^2-a^2}.$$

Example 1. Find $\int \frac{dx}{\sqrt{a^2-x^2}}$.

Solution. Let $x = a \operatorname{Sin} \theta$. Then $dx = a \operatorname{Cos} \theta \, d\theta$.

$$\text{Now } \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a \operatorname{Cos} \theta \, d\theta}{a \operatorname{Cos} \theta} = \int d\theta = \theta + c$$

$$= \operatorname{Sin}^{-1} \frac{x}{a} + C.$$

Example 2. Evaluate $\int_{\frac{-a}{2}}^{\frac{a\sqrt{3}}{2}} \sqrt{a^2-x^2}\, dx$

Solution. Let $x = a \operatorname{Sin} \theta$ (or $\theta = \operatorname{arc} \operatorname{Sin} \frac{x}{a}$).

Then $dx = a \operatorname{Cos} \theta \, d\theta$. So that

$\sqrt{a^2-x^2} = a \operatorname{Cos} \theta$.

$$\theta = \operatorname{arc} \operatorname{Sin} \left(-\frac{1}{2}\right)$$

$$= -\frac{\pi}{6} \text{ when } x = -\frac{a}{2}$$

$$\theta = \operatorname{arc} \operatorname{Sin}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

$$\text{when } x = \frac{a\sqrt{3}}{2}.$$

$$\text{Thus } \int_{\frac{-a}{2}}^{\frac{a\sqrt{3}}{2}} \sqrt{a^2-x^2}\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} a^2 \operatorname{Cos}^2 \theta \, d\theta$$

$$= \frac{a^2}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 + \operatorname{Cos} 2\theta) \, d\theta$$

$$= \left[\frac{a^2}{2}\left(\theta + \frac{\operatorname{Sin} 2\theta}{2}\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\frac{a^2}{2}\left[\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)-\left(-\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)\right]$$

$$=\frac{a^2}{2}\left(\frac{\pi+\sqrt{3}}{2}\right).$$

Example 3. Find $\int x^3\sqrt{7+x^2}\,dx$.

Solution. Let $x=\sqrt{7}\operatorname{Tan}\theta$, then $dx=\sqrt{7}\operatorname{Sec}^2\theta\,d\theta$.

So that $\sqrt{7+x^2}=\sqrt{7+7\operatorname{Tan}^2\theta}=\sqrt{7}\operatorname{Sec}\theta$.

Now $\int x^3\sqrt{7+x^2dx}$

$$=\int 7^{\frac{3}{2}}\operatorname{Tan}^3\theta\,\sqrt{7}\operatorname{Sec}\theta\,\sqrt{7}\operatorname{Sec}^2\theta\,d\theta$$

$$=7^{\frac{5}{2}}\int\operatorname{Tan}^3\theta\operatorname{Sec}^3\theta\,d\theta$$

$$=7^{\frac{5}{2}}\int(\operatorname{Sec}^2\theta-1)\operatorname{Sec}^2\theta(\operatorname{Sec}\theta\operatorname{Tan}\theta)\,d\theta$$

$$=7^{\frac{5}{2}}\int(\operatorname{Sec}^4\theta-\operatorname{Sec}^2\theta)\,d\operatorname{Sec}\theta$$

$$=7^{\frac{5}{2}}\left(\frac{\operatorname{Sec}^5\theta}{5}-\frac{\operatorname{Sec}^3\theta}{3}\right)+c$$

$$=\frac{1}{5}\left(7+x^2\right)^{\frac{5}{2}}-\frac{7}{3}\left(7+x^2\right)^{\frac{3}{2}}+c.$$

Example 4. Evaluate $\int_{-6}^{-3}\frac{\sqrt{x^2-9}}{x}dx$.

Solution. We have $a=3$ and must choose θ so that $x=3\operatorname{Sec}\theta$ is negative, we therefore use the second range *i.e*

$-\pi\leqslant\theta<-\frac{\pi}{2}$. For $x=-6$, $\operatorname{Sec}\theta=-2$,

$$\theta=-\frac{2\pi}{3}.$$

For $x=-3$, $\operatorname{Sec}\theta=-1$ or $\theta=-\pi$.

Also, $dx = 3\ \text{Sec}\ \theta\ \text{Tan}\ \theta\ d\theta$; $\sqrt{x^2-9} = 3\ \text{Tan}\ \theta$.

$$\text{Now} \int_{-6}^{-3} \frac{\sqrt{x^2-9}}{x}dx = \int_{-\frac{2\pi}{3}}^{-\frac{\pi}{3}} \frac{3\ \text{Tan}\ \theta}{3\ \text{Sec}\ \theta}\,3\ \text{Sec}\ \theta\ \text{Tan}\ \theta\ d\theta$$

$$= 3\int_{\frac{-2\pi}{3}}^{-\pi} \text{Tan}^2\ \theta\ d\theta = \Big[3\ \big(\text{Tan}\ \theta - \theta\ \big)\Big]_{\frac{-2\pi}{3}}^{-\pi}$$

$$= 3\Big[(\theta+\pi) - \Big(\sqrt{3} + \frac{2\pi}{3}\Big)\Big] = \pi - 3\sqrt{3}.$$

Exercise 5.5

Evaluate the following.

(*i*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

(*ii*) $\displaystyle\int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2-9}}$

(*iii*) $\displaystyle\int_0^2 \frac{x^2\,dx}{\sqrt{x^2+4}}$

(*iv*) $\displaystyle\int_0^2 \frac{x^3\,dx}{\sqrt{16-x^2}}$

(v) $\int_0^{2\sqrt{3}} \frac{x^3\,dx}{\sqrt{x^2+4}}$

(vi) $\int_0^1 \frac{x^3\,dx}{\sqrt{4-x^2}}$

2. Evaluate the following :

(i) $\int \frac{2dx}{x\sqrt{x^2-5}}$

(ii) $\int \frac{x^3\,dx}{\sqrt{a^2-x^2}}$

iii) $\int \frac{x^3\,dx}{\sqrt{2x^2+7}}$

(iv) $\int u^3\sqrt{a^2u^2+b^2}\,du$

(v) $\int \frac{\sqrt{x^2-a^2}}{x}\,dx$

(vi) $\int \frac{dx}{x^2\sqrt{a^2-x^2}}$

(vii) $\int \frac{dx}{x^2\sqrt{x^2+a^2}}$

(viii) $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}}$

(ix) $\int \frac{dx}{x^4\sqrt{a^2-x^2}}$

3. Evaluate :—

(i) $\int_0^a \frac{dx}{(a^2+x^2)^{\frac{3}{5}}}$

(ii) $\displaystyle\int_0^1 \frac{x^2dx}{(4-x^2)^{\frac{3}{2}}}$

(iii) $\displaystyle\int_{\sqrt{3}}^{3\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+9}}$

$\displaystyle\int_0^{\sqrt{5}} x^2\sqrt{5-x^2}\,dx.$

ANSWERS

1. (i) $\frac{\pi}{6}$

   (ii) $-\frac{\pi}{18}$

   (iii) $2\sqrt{2}-2\ ln\,(\frac{1}{2}+\sqrt{2})$

   (iv) $16\left[\frac{8}{3}-\frac{3\sqrt{3}}{2}\right]$

   (v) $\frac{32}{3}$

   (vi) $\frac{16-9\sqrt{3}}{3}$

2. (i) $\frac{2}{\sqrt{5}}$ arc Sec $\frac{x}{\sqrt{5}}+c$

   (ii) $\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}-a^2\sqrt{a^2-x^2}+c$

   (iii) $\frac{1}{12}(2x^2+7)^{\frac{3}{2}}-\frac{7}{4}\sqrt{2x^2+7}+c$

(iv) $(\frac{1}{5}a^4)(a^2u^2+b^2)^{\frac{5}{2}}-\left(\frac{b^4}{3a^4}\right)\left(a^2u^2+b^2\right)^{\frac{3}{2}}+c$

(v) $\sqrt{x^2-a^2}-a \text{ arc Sec}\left(\frac{x}{a}\right)+c$

(vi) $-\frac{1}{a^2}\frac{\sqrt{a^2-x^2}}{x}+c$

(vii) $-\frac{\sqrt{x^2+a^2}}{a^2x}+c$

(viii) $\frac{1}{a^2}\cdot\frac{x}{\sqrt{a^2-x^2}}+c$

(ix) $\frac{1}{a^4}\left[-\frac{(a^2-x^2)^{\frac{3}{2}}}{3x^3}-\frac{\sqrt{a^2-x^2}}{x}\right]+c$

3. (i) $\frac{1}{a^2\sqrt{2}}$

(ii) $3^{-\frac{1}{2}}-\frac{\pi}{6}$

(iii) $\frac{1}{9}\left(2-\frac{2}{\sqrt{3}}\right)$

(iv) $\frac{25\pi}{16}$.

5.8 *Completing the Square.* If an integrand involves h quadratic function in the denommator or under a radical, the important device is to complete the square in the quadratic expression.

Example 1. Find $\int\frac{dx}{\sqrt{6x-4x^2}}$.

Solution. $6x-4x^2=-\left(4x^2-\frac{3}{2}x+\frac{9}{16}\right)+\frac{9}{4}$

$$=\frac{9}{4}-4(x-\tfrac{3}{4})^2.$$

Thus $\int\frac{dx}{\sqrt{6x-4x^2}}=\int\frac{dx}{\sqrt{\frac{9}{4}-4(x-\frac{3}{4})^2}}$

Let $y=2\left(x-\frac{3}{4}\right)$. Then $dy=2\,dx$ and

$$\int \frac{dx}{\sqrt{6x-4x^2}}=\frac{1}{2}\int \frac{ds}{\sqrt{\frac{9}{4}-y^2}}=\frac{1}{2}\,\text{Sin}^{-1}\frac{y}{\frac{3}{2}}+c$$

$$=\frac{1}{2}\,\text{Sin}^{-1}\frac{16x-3}{12}+c.$$

Example 2. Find $\int \frac{2x-3}{x^2+2x+2}dx$.

Solution. $x^2+2x+2=(x+1)^2+1$.

Thus $\int \frac{(2x-3)\,dx}{x^2+2x+2}=\int \frac{(2x-3)\,dx}{(x+1)^2+1}$.

Let $y=x+1$. Then $dy=dx$ and

$$\int \frac{2x-3}{(x+1)^2+1}dx=\int \frac{2y\,dy}{y^2+1}$$

$$=\int \frac{2y\,dy}{y^2+1}-5\int \frac{dy}{y^2+1}$$

$$=\int \frac{d\,(y^2+1)}{y^2+1}-5\int \frac{dy}{y^2+1}$$

$$=\ln\,(y^2+1)-5\text{ arc Tan }y+c$$

$$=\ln\,(x^2+2x+2)-5\text{ arc Tan }(x+1)+c.$$

Exercise 5.6

Calculate the following integrals :—

1. $\int \frac{dx}{x^2+4x+5}$
2. $\int \frac{dx}{\sqrt{5+4x-x^2}}$
3. $\int \frac{dx}{x^2-x+1}$
4. $\int \frac{dx}{x^2+6x+8}$

5. $\int \frac{dx}{4x-x^2}$

6. $\int \frac{ds}{\sqrt{4s-s^2}}$

7. $\int \frac{dx}{(x+2)\sqrt{x^2+4x+3}}$

8. $\int \frac{dx}{(x-1)\sqrt{x^2-2x}}$

9. $\int \frac{dx}{\sqrt{5-2x+x^2}}$

10. $\int \frac{(2x-5)\,dx}{\sqrt{4x-x^2}}$

11. $\int \frac{(x+3)\,dx}{x^2+2x+5}$

12. $\int \frac{(x+2)\,dx}{\sqrt{x^2+2x-3}}$

13. $\int \frac{(2-x)\,dx}{4x^2+4x-3}$

14. $\int \sqrt{3-2x-x^2}\,dx$

15. $\int \sqrt{4x^2-4x+5}\,dx$

16. $\int \sqrt{x^2-8x}\,dx.$

ANSWERS

1. $\text{arc Tan}\,(x+2)+c$

2. $\text{arc Sin}\left(\frac{x-2}{3}\right)+c$

3. $\frac{2}{\sqrt{3}}\,\text{arc Tan}\,\frac{2x-1}{\sqrt{3}}+c$

4. $\frac{1}{2}\, ln\left(\frac{x+2}{x+4}\right)+c$

5. $\frac{1}{4}\, ln\left(\frac{x}{4-x}\right)+c$

6. $ln\,(s+2+\sqrt{4s+s^2})+c$

7. arc Sec $(x+2)+c$

8. $\frac{1}{2}$ arc Sec $\frac{x-1}{2}+c$

9. $ln\,(x-1+\sqrt{x^2-2x+5})+c$

10. $-2\sqrt{4x-x^2}-\text{Sin}^{-1}\frac{x-2}{2}+c$

11. $\frac{1}{2}\, ln\,(x^2+2x+5)+$ arc Tan$\left(\frac{x+1}{2}\right)+c$

12. $\sqrt{x^2+2x-3}+ln(x+1+\sqrt{x^2+2x-3})+c$

13. $-\frac{1}{8}\, ln\,(4x^2+4x-3)+\frac{5}{16}\, ln\left(\frac{2x-1}{2x+3}\right)+c$

14. $\frac{x+1}{2}\sqrt{3-2x-x^2}+2$ arc Sin $\frac{x+1}{2}+c$

15. $\frac{2x-1}{4}\sqrt{4x^2-4x+5}+ln(2x-1+\sqrt{4x^2-4x+5})+$

16. $\frac{1}{2}\,(x-4)\sqrt{x^2-8x}-8\, ln\,(x-4+\sqrt{x^2-8x})+c$

5.9 *Integration by Parts.* From the formula

$$d\,(uv)=udv+vdu \qquad (1)$$

we obtain $uv=\int udv+\int v\,du.$ (2)

From (2) we obtain

$$\int udv=uv-\int vdu \qquad (3)$$

= First function × integration of

2nd function $-\int$ derivative of first × integration of 2nd function.

which is known as the formula for integration by parts.

If a given differential can be written as a product udv n such a way that the product can be integrated, we can use this formula.

Example 1 Find $\int x \operatorname{Sin} x \, dx$.

Solution. Let $u = x$, $dv = \operatorname{Sin} x \, dx$. Then $du = dx$ and $v = -\operatorname{Cos} x$.

Now (3) gives

$$\int x \operatorname{Sin} x \, dx = -x \operatorname{Cos} x + \int \operatorname{Cos} x \, dx$$
$$= -x \operatorname{Cos} x + \operatorname{Sin} x + c.$$

Example 2. Find $\int \ln x \, dx$.

Solution. Let $u = \ln x$, $dv = dx$.

Then $du = \dfrac{dx}{x}$, $v = x$.

$$\therefore \int \ln x \, dx = x \ln x - \int x \frac{dx}{x}$$
$$= x \ln x - x + c.$$

Example 3. Find $\int x^2 e^x \, dx$.

Solution. Let $u = x^2$, $dv = dx$

Then $du = 2x \, dx$, $v = e^x$.

$$\therefore \int x^2 e^x dx = x^2 e^x - 2 \int x \, e^x dx.$$

Now let $u_1 = x$, $dv_1 = e^x \, dx$ in the second integral. We have,

$$\int x^2 e^x \, dx = x^2 e^x - 2 \left[x \, e^x - \int e^x \, dx \right]$$
$$= x^2 e^x - 2x \, e^x + 2e^x + c$$
$$= e^x(x^2 - 2x + 2) + c.$$

Example 4. Find $\int e^{ax} \operatorname{Sin} bx \, dx$.

Solution. Let $u = e^{ax}$, $dv = \operatorname{Sin} bx \, dx$.

Then $du = a\,e^{x}\,dx, \quad v = -\frac{1}{v}\text{Cos } bx.$

$$\therefore \int e^{ax}\text{ Sin } bx\,dx = -\frac{1}{b}e^{ax}\text{ Cos } bx + \frac{a}{b}\int e^{ax}\text{ Cos } bx\,dx.$$

In the second Integral, let $u = e^{ax}$, $dv = \text{Cos } bx\,dx$.

Then $du = a\,e^{ax}\,dx$, $v = \frac{1}{b}\text{ Sin } bx.$

$$\text{Thus} \quad \int e^{ax}\text{ Cos } bx\,dx = \frac{1}{b}e^{ax}\text{ Sin } bx - \frac{a}{b}\int \text{Sin } bx\,\text{d}x.$$

$$\text{Hence} \int e^{ax}\text{ Sin } bx\,dx = -\frac{1}{b}e^{ax}\text{ Cos } bx$$

$$+\frac{a}{b}\left[\frac{1}{b}e^{ax}\text{ Sin } bx - \frac{a}{b}\int e^{ax}\text{ Sin } bx\,dx\right]$$

$$\text{or}\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\text{ Sin } bx\,dx = -\frac{1}{b}e^{ax}\text{ Cos } bx + \frac{a}{b^2}e^{ax}\text{ Sin } bx$$

$$i.e. \int e^{ax}\text{ Sin } bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\text{ Sin } bx - b\text{ Cos } bx\right).$$

This gives us one indefinite integral of e^{ax} Sin bx. To get the complete answer we add c on the right.

(Theorm) 5.10. $\int e^{ax}[a\,f(x)+f'(x)]dx = e^{ax}f(x)+c.$

The proof follows from the fact that

$$\frac{d}{dx}\left[e^{ax}f(x)+c\right] = \frac{d}{dx}\left[e^{ax}f(x)\right]; \quad \left(\because \frac{dc}{dx} = 0\right)$$

$$= e^{ax}f'(x) + f(x)\,a\,e^{ax}$$

$$= e^{ax}\left[a\,f(x) + f'(x)\right].$$

Hence the result.

Example 1. Evaluate $\int e^{x}\left\{\frac{1}{1-x}+\frac{1}{(1-x)^2}\right\}dx.$

Solution. Since $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$,

thus $\int e^x \left\{\frac{1}{1-x}+\frac{1}{(1-x)^2}\right\} dx = \frac{e^x}{1-x}+c.$

Exercise 5.7

1. *Integrate by parts :*

(*i*) $\int x \ln x \, dx$

(*ii*) $\int x \operatorname{Sin} x \, dx$

(*iii*) $\int x^2 \operatorname{Tan}^{-1} x \, dx$

(*iv*) $\int x^3 \operatorname{Tan}^{-1} x \, dx$

(*v*) $\int x^2 \ln x \, dx$

(*vi*) $\int \operatorname{Tan}^{-1} x \, dx$

(*vii*) $\int x \operatorname{Sin}^{-1} x \, dx$

(*viii*) $\int x^2 \operatorname{Sin}^{-1} x \, dx$

(*ix*) $\int \ln (x+\sqrt{x^2+1}) \, dx$

·2. *Evaluate :*

(*i*) $\int \sqrt{4-x^2} \, dx$

(*ii*) $\int \sqrt{4-5x^2} \, dx$

(*iii*) $\int \sqrt{a^2 - x^2}\, dx.$

3. Find the following indefinite integrals :—

(*i*) $\int (ln\, x)^2\, dx$

(*ii*) $\int x^3 e^{2x}\, dx$

(*iii*) $\int \text{Sin}^{-1} x\, dx$

(*iv*) $\int x\, \text{Tan}^{-1} x\, dx$

(*v*) $\int x\, \text{Sec}^{-1} x\, dx$

(*vi*) $\int x\, \text{Cosec}^2 \frac{x}{2}\, dx$

(*vii*) $\int \frac{2x\, dx}{\text{Cos}^2 2x}$

(*viii*) $\int 9x\, \text{Tan}^2 3x\, dx$

(*ix*) $\int \text{Sin}\sqrt{2x}\, dx$

(*x*) $\int 6x^2 \text{ arc Sin } 2x\, dx$

(*xi*) $\int x^m\, ln\, x\, dx\ ;\ m \neq -1$

(*xii*) $\int 2x^3 e^{x^2}\, dx$

(*xiii*) $\int e^x\, \text{Sin}\, 2x\, dx$

(*xiv*) $\int e^{3x}\, \text{Sin}\, 2x\, dx$

(*xv*) $\int e^{-x} \operatorname{Cos} 3x \, dx$

(*xvi*) $\int e^{ax} \operatorname{Cos} bx \, dx.$

4. Calculate the following :—

(*i*) $\int e^{x} \frac{1+\operatorname{Sin} x}{1+\operatorname{Cos} x} \, dx$

(*ii*) $\int \frac{xe^{x}}{(1+x)^2} \, dx$

(*iii*) $\int \frac{e^{x} (1+x)}{(2+x)^2} \, dx$

(*iv*) $\int \operatorname{Cos} \left(b \log \frac{x}{a}\right) dx$

(*v*) $\int e^{x} \operatorname{Sin} mx \operatorname{Cos} nx \, dx$

(*vi*) $\int e^{x} (\operatorname{Sin} x+\operatorname{Cos} x) \, dx$

(*vii*)) $\int e^{x} \left(\operatorname{Sec}^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right) dx$

ANSWERS

(*i*) $\frac{x^2}{2} \ln x - \frac{x^2}{4} + c$

(*ii*) $-x \operatorname{Cos} x+\operatorname{Sin} x+c$

(*iii*) $\frac{x^3}{3} \operatorname{Tan}^{-1}x- \frac{x^2}{6} + \frac{1}{6} \mathit{in} (1+x^2) + c$

(*iv*) $\frac{x^4-1}{4} \operatorname{Tan}^{-1} x - \frac{x^3}{12} + \frac{x}{4} + c$

(*v*) $\frac{x^3}{3} \ln x - \frac{x^3}{9} + c$

(*vi*) $x \,\text{Tan}^{-}x \; \frac{1}{2} \ln (1+x^2)+c$

(*vii*) $\frac{1}{4}(2x^2-1)\,\text{Sin}^{-1} x + \frac{x\sqrt{1-x^2}}{4} + c$

(*viii*) $\frac{x^3}{3}\,\text{Sin}^{-1}x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{\frac{3}{2}} + c$

(*ix*) $x \ln (x+\sqrt{x^2+1}) - \sqrt{x^2+1} + c$

2. (*i*) $\frac{x\sqrt{4-x^2}}{2} + 2\,\text{Sin}^{-1}\frac{x}{2} + c$

(*ii*) $\frac{2}{\sqrt{5}}\,\text{arc Sin}\,\frac{\sqrt{5}\,x}{2} + \frac{x\sqrt{4-5x^2}}{2} + c.$

(*iii*) $\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\,\text{Sin}^{-1}\frac{x}{a} + c$

3. (*i*) $x(\ln x)^2 - 2x \ln x + 2x + c$

(*ii*) $e^{2x}\left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8}\right) + c$

(*iii*) $x\,\text{Sin}^{-1} x + \sqrt{1-x^2} + c$

(*iv*) $\frac{x^2}{2}\,\text{Tan}\quad x - \frac{x}{2} + \frac{1}{2}\,\text{Tan}^{-1} x + c$

(*v*) $\frac{x^2}{2}\,\text{Sec}^{-1} x - \frac{1}{2}\sqrt{x^2-1} + c$

(*vi*) $-2x\,\text{Cot}\left(\frac{x}{2}\right) + 4 \ln\left(\text{Sin}\,\frac{x}{2}\right) + c$

(*vii*) $x\,\text{Tan}\,2x + \frac{1}{2} \ln \text{Cos}\,2x + c$

(*viii*) $3x\,\text{Tan}\,3x - \frac{9}{2}x^2 + \ln \text{Cos}\,3x + c$

(*ix*) $-\sqrt{2x}\,\text{Cos}\,\sqrt{2x} + \text{Sin}\,\sqrt{2x} + c$

(x) $2\,x^3 \text{ arc Sin } 2x + \frac{1}{4}\sqrt{1-4x^2} - \frac{1}{12}(1-4x^2)^{\frac{3}{2}} + c$

(xi) $\frac{x^{m+1}\ln x}{m+1} - \frac{x^{m+1}}{(m+1)^2} + c$

(xii) $(x^2-1)\,e^{x^2} + c$.

(xiii) $\frac{1}{5}\,e^x\,(\text{Sin } 2x - 2 \text{ Cos } 2x) + c$

(xiv) $\frac{1}{13}e^{3x}\,(3 \text{ Sin } 2x - 2 \text{ Cos } 2x) + c$

(xv) $e^{-x}\,\frac{1}{10}\,(3 \text{ Sin } 3x - \text{Cos } 3x) + c$

(xvi) $\frac{e^{ax}}{a^2+b^2}\,(a \text{ Cos } bx + b \text{ Sin } bx) + c$

4. (i) $e^x \text{ Tan}\frac{x}{2} + c$

(ii) $\frac{e^x}{1+x} + c$

(iii) $\frac{e^x}{x+2} + c$

(iv) $\frac{x}{\sqrt{1+b^2}}\left[\text{Cos}\left(b \log\frac{x}{a}\right) + \text{b Sin}\left(b \log\frac{x}{a}\right)\right] + c$

(v) $\frac{1}{2}\frac{1}{\sqrt{(m+n)^2+1}}\,e^x \text{ Sin}\{(m+n)\,x - \text{Tan}^{-1}\,(m+n)\}$

$+\frac{1}{2}\,\frac{1}{\sqrt{(m-n)^2+1}}e^x \text{ Sin }\{(m-n)x - \text{Tan}^{-1}\,(m-n)\}$

(vi) $e^x \text{ Sin } x + c$

(vii) $e^x \text{ Sec}^{-1}\, x + c$

5.11. *Integration of Rational Functions.* Using the methods of Chapter XI of Modern Algebra and Trigonometry, the rational function can be resolved into algebraic sum of partial fractions and integrated as illustrated in the following examples.

Example 1. Find

(*i*) $\int \frac{dx}{x^2-a^2}, \; x>a$

(*ii*) $\int \frac{dx}{a^2-x^2}, \; x<a$

Solution.

(*i*) $\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}=\frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right].$

$$\text{Thus} \int \frac{dx}{x^2-a^2}=\frac{1}{2a}\int\frac{dx}{x-a}-\frac{1}{2a}\int\frac{dx}{x+a}$$

$$=\frac{1}{2a}\, ln\,(x-a)-\frac{1}{2a}\, ln\,(x+a)+c$$

$$=\frac{1}{2a} ln\left(\frac{x-a}{x+a}\right)+c.$$

(*ii*) $\frac{1}{a^2-x^2}=\frac{1}{(a+x)(a-x)}=\frac{1}{2a}\left[\frac{1}{a+x}+\frac{1}{a-x}\right].$

$$\text{Thus} \int \frac{dx}{a^2-x^2}=\frac{1}{2a}\int\frac{dx}{a+x}+\frac{1}{2a}\int\frac{dx}{a-x}$$

$$=\frac{1}{2a}\, ln\,(a+x)-\frac{1}{2a}\, ln\,(a-x)$$

$$=\frac{1}{2a}\left(\frac{a+x}{a-x}\right)+c.$$

Example 2. Find $\int \frac{2x-1}{x(x-1)(x-3)}\,dx.$

We know that $\frac{2x-1}{x(x-1)(x-3)}=\frac{1}{3}\cdot\frac{1}{x}-\frac{1}{2}\cdot\frac{1}{x-1}+\frac{5}{6}\cdot\frac{}{x}$

(By partial fractions

Hence $\int \frac{2x-1}{x(x-1)(x-3)}\,dx$

$$= -\tfrac{1}{3}\int \frac{1}{x}\,dx - \tfrac{1}{2}\int \frac{dx}{x-1} + \tfrac{5}{6}\int \frac{dx}{x-3}$$

$$= -\tfrac{1}{3}\ln x - \tfrac{1}{2}\ln(x-1) + \tfrac{5}{6}\ln(x-3) + c.$$

Example 3. Find $\int \frac{x^3-2x^2}{x^2+y}\,dx.$

Solution. $\int \frac{x^3-2x^2}{x^2+9}\,dx = \int \left[x-2+\frac{-9x+18}{x^2+9}\right] lx.$

Hence $\int \frac{x^3-2x^2}{x^2+9}\,dx = \int (x-2)\,dx - 9\int \frac{x\,dx}{x^2+9}$

$$+18\int \frac{dx}{x^2+9}$$

$$= \frac{x^2}{2} - 2x - \tfrac{9}{2}\ln(x^2+9) + 6\,\text{Tan a}\,\frac{x}{3} + c.$$

Example 4. Find $\int \frac{(x+5)\,dx}{(x-1)^2(x+2)}$

Solution. Partial fractions of $\frac{x+5}{(x-1)^2(x+2)}$ are

$$\frac{2}{(x-1)^2} - \frac{1}{3(x-1)} + \frac{1}{3(x+2)}.$$

Hence $\int \frac{(x+5)dx}{(x-1)^2(x+2)} = 2\int \frac{dx}{(x-1)^2} - \tfrac{1}{3}\int \frac{dx}{x-1} + \tfrac{1}{3}\int \frac{dx}{x+2}$

$$= -\frac{2}{x-1} - \tfrac{1}{3}\ln(x-1) + \tfrac{1}{3}\ln(x+2) + c.$$

Example 5. Find $\int \frac{3x^2+x-2}{(x-1)(x^2+1)}\,dx.$

Solution. Partial fractions of $\frac{3x^2+x-2}{(x-1)(x^2+1)}$ are

$$\frac{1}{x-1} + \frac{2x+3}{x^2+1}.$$

Hence $\int \frac{3x^2+x-2}{(x-1)(x^2+1)}\,dx = \int \frac{dx}{x-1} + \int \frac{2x+3}{x^2+1}\,dx$

$$= \int \frac{dx}{x-1} + \int \frac{d(x^2+1)}{x^2+1} + 3\int \frac{dx}{x^2+1}$$

$$= \ln(x-1) + \ln(x^2+1) + 3 \text{ arc Tan } x + c.$$

Example 6. Find $\int \frac{x^2dx}{1-x^6}$.

Soluntion. Let $x^3 = t$, then $3x^2\,dx = dt$

$$\therefore \int \frac{x^2\,dx}{1-x^6} = \frac{1}{2}\int \frac{dt}{1-t^2} = \frac{1}{3}\ln\left(\frac{1+t}{1-t}\right) + c$$

$$= \frac{1}{3}\ln\left(\frac{1+x^3}{1-x^3}\right) + c.$$

Exercise 5.8

Calculate the following integrals :

1. $\int \frac{dx}{x^2-25}$
2. $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}\,dx$
3. $\int \frac{(2x-1)\,dx}{x(x-1)(x-3)}$
4. $\int \frac{dx}{2x^2-5x+2}$
5. $\int \frac{\text{Cos } x\,dx}{\text{Sin } x\,(2+\text{Sin } x)}$
6. $\int_0^{\frac{\pi}{2}} \frac{\text{Sin } x\,dx}{(1+\text{Cos } x)(2+\text{Cos } x)}$

7. $\int \frac{x^2+3x+4}{x-2}\,dx$

8. $\int \frac{x^3-x^2+2x+3}{x^2+3x+2}\,dx$

9. $\int \frac{2x\,dx}{(1+x^2)(3+x^2)}$

10. $\int \frac{x^2+2x+3}{x^2-3x+2}\,dx$

11. $\int \frac{x^4+1}{x^3-x}\,dx$

12. $\int \frac{x^3+3}{x^2+4}\,dx$

13. $\int \frac{(x^2-2)}{(x+1)(x-1)^2}\,dx$

14. $\int \frac{x^2+3x+3}{(x+1)(x^2+1)}\,dx$

15. $\int \frac{x^2-2x-3}{(x-1)(x^2+2x+2)}\,dx$

16. $\int \frac{x-3}{(x+1)^2(x-2)}\,dx$

17. $\int \frac{x^2+1}{(x-1)^3}\,dx$

18. $\int \frac{2x^2-1}{(x+1)^2(x-3)}\,dx$

19. $\int \frac{x^3+1}{(x^2-1)^2}\,dx$

20. $\int \frac{x^3-2x^2+3x-4}{(x-1)^2(x^2+2x+2)}\,dx$

21. $\int \frac{x^2-3x+5}{x^4-8x^2+16}\,dx$

22. $\int \frac{x^2+3x+5}{x^3+8}\,dx$

ANSWERS

1. $\frac{1}{10} \ln \left(\frac{x-5}{x+5}\right) + c$
2. $\ln (x-1)(x-2)(x-3) + c$
3. $\ln \frac{(x-3)^{\frac{5}{6}}}{x^{\frac{1}{3}} (x-1)^{\frac{1}{2}}} + c$
4. $\frac{1}{3} \ln \left(\frac{x-2}{2x-1}\right) + c$
5. $\frac{1}{2} \ln \text{Sin}\, x - \frac{1}{2} \ln (2 + \text{Sin}\, x) + c$
6. $\ln \frac{4}{3}$
7. $\frac{x^2}{2} + 5x + 14 \ln (x-2) + c$
8. $\frac{1}{2} x^2 - 4x + 13 \ln (x+2) - \ln (x+1) + c$
9. $\frac{1}{2} \ln \frac{1+x^2}{3+x^2} + c$
10. $x + 11 \ln (x-2) - 6 \ln (x-1) + c$
11. $\frac{x^2}{2} - \ln x + \ln (x-1) + \ln (x+1) + c$
12. $\frac{x^2}{2} - 2 \ln (x^2+4) + \frac{3}{2} \text{arc Tan} \frac{x}{2} + c$
13. $-\frac{1}{4} \ln (x+1) + \frac{5}{4} \ln (x-1) + \frac{1}{2(x-1)} + c$
14. $\frac{1}{2} \ln (x+1) + \frac{1}{4} \ln (x^2+1) + \frac{5}{2} \text{arc Tan}\, x + c$
15. $-\frac{4}{5} \ln (x-1) + \frac{9}{10} \ln (x^2+2x+2) - \frac{2}{5} \text{arc Tan} (x+1) + c$
16. $-\frac{4}{3} (x+1)^{-1} + \frac{1}{9} \ln (x+1) - \frac{1}{9} \ln (x-2) + c$
17. $\ln (x-1) - \frac{2}{x-1} - \frac{1}{(x-1)^2} + c$

18. $\frac{17}{16} ln\,(x-3)+\frac{15}{16} ln\,(x+1)+\frac{1}{4\,(x+1)}.$

19. $\frac{3}{4} ln\,(x+1)+\frac{1}{4} ln\,(x-1)-\frac{1}{2(x+1)}$

20. $\frac{2}{5}(x-1)^{-1}+\frac{18}{25} ln\,(x-1)+\frac{7}{50} ln\,(x^2+2x+2)$

$-\frac{51}{25}\,\mathrm{Tan}^{-1}\,(x+1)+c$

21 $-\frac{3}{16}(x-2)^{-1}-\frac{1}{32} ln\,(x-2)-\frac{15}{16}(x+2)^{-1}$

$+\frac{1}{32} ln\,(x+2)+c$

22. $\frac{1}{4} ln\,(x+2)+\frac{3}{8} ln\,(x^2-2x+4)+$

$\left(\frac{11\sqrt{3}}{12}\right)\text{arc Tan}\left(\frac{x-1}{\sqrt{3}}\right)+c.$

CHAPTER VI

HIGHER DERIVATIVES AND EXPANSION OF FUNCTIONS

6.1. *Derivatives of Second and Higher Order.* We have noticed in the previous chapters that the derivative $f'(x)$ of a function $f(x)$ is in general, itself a function of x and so it can again be differentiated with respect to x. The derivative of $f'(x)$ is said to be the second derivatives of $f'(x)$ and is denoted by $f''(x)$. Again the derivative of $f''(x)$ is known as the third derivative of $f(x)$ and is denoted by $f'''(x)$. The fourth, fifth..........., nth derivatives of $f(x)$ are similarly denoted by $f^4(x)$, $f^5(x)$,...................., $f^n(x)$ respectively.

If $y = f(x)$, then we know that $\frac{dy}{dx} = f'(x)$. Thus the successive derivatives of y with respect to x are denoted by $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$,.............., $\frac{d^ny}{dx^n}$

$\frac{d^2y}{dx^2}$ is read as " d two y by dx squared" and so on for $\frac{d^3y}{dx^3}$,.............., $\frac{d^ny}{dx^n}$. It is to be noted that $\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \forall n = 1, 2, 3.........n$. Sometimes instead of writing $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$,.............., $\frac{d^ny}{dx^n}$ we write y_1, y_2, y_3,..........., y_n.

The nth derivative of $f(x)$ is also called the derivative of nth order.

The symbols $f^n(a)$, $\left[\frac{d^ny}{dx^n}\right]_{x=a}$ or $y_n(a)$ denote the value of the nth derivative of $y=f(x)$ at $x=a$. In other worde

$$\left[\, f^n(x) \,\right]_{x=a} = f^n(a) = \left[\frac{d^ny}{dy^n}\right]_{x=a} = y_n(a)$$

$$\forall n = 0, 1, 2, \ldots\ldots\ldots, n;$$

where, of course, $f^0(x)$ means $f(x)$ itself.

Example 1 Find y_3, when $y=x^7$.

Solution. $y=x^7 \Rightarrow y_1=7x^6 \Rightarrow y_2=42x^5$

$$\Rightarrow y_3=210x^4.$$

Example 2. Find $y_4(0)$, when $y=\frac{x^4}{x+1}$.

Solution. By actual division, we have

$$y=\frac{x^4}{x+1}=x^3-x^2+x-1+\frac{1}{x+1},$$

$$=x^3-x^2+x-1+(x+1)^{-1}$$

$$\Rightarrow y_1=3x^2-2x+1+(-1)(x+1)^{-2}$$

$$\Rightarrow y_2=6x-2+(-1)(-2)(x+1)^{-3}$$

$$\Rightarrow y_3=6+(-1)(-2)(-3)(x+1)^{-4}$$

$$\Rightarrow y_4=(-1)(-2)(-3)(-4)(x+1)^{-5}$$

$$=\frac{24}{(x+1)^5}$$

$$\Rightarrow y_4(0) = \frac{24}{(0+1)^5} = 24.$$

Example 3. Find the second derivative of x^n Sin x.

Solution. Let $f(x)=x^n \text{ Sin } x$.

Then $f'(x)=x^n \text{ Cos } x+nx^{n-1} \text{ Sin } x$,

and $f''(x)=-x^n \text{ Sin } x+nx^{n-1} \text{ Cos } x+n\{x^{n-1} \text{ Cos } x$

$$+(n-1)\,x^{n-2} \text{ Sin } x\}$$

$$=\{n(n-1)\,x^{n-2}-x^n\}\text{Sin } x+2\,nx^{n-1} \text{ Cos } x.$$

Example 4. If $x=a\,(\theta+\text{Sin}\,\theta)$, $y=a\,(1-\text{Cos}\,\theta)$

Show that $\left[\dfrac{d^2y}{dx^2}\right]_{x=\frac{\pi}{2}}$ is $\dfrac{1}{a}$.

Solution. $x=a\,(\theta+\text{Sin}\,\theta)$, $y=a\,(1-\text{Cos}\,\theta)$

$$\Rightarrow \frac{dx}{d\theta}=a(1+\text{Cos}\,\theta),\ \frac{dy}{d\theta}=a\,\text{Sin}\,\theta$$

$$\Rightarrow \frac{dy}{dx}=\frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}=\frac{\text{Sin}\,\theta}{1+\text{Cos}\,\theta}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\text{Sin}\,\theta}{1+\text{Cos}\,\theta}\right)$$

$$=\frac{(1+\text{Cos}\,\theta)\left(\left(\text{Cos}\,\theta\,\dfrac{d\theta}{dx}\right)-\text{Sin}\,\theta\left(-\text{Sin}\,\theta\,\dfrac{d\theta}{dx}\right)\right.}{(1+\text{Cos}\,\theta)^2}$$

$$=\frac{(1+\text{Cos}\,\theta)\left(\text{Cos}\,\theta\,\dfrac{1}{a(1+\text{Cos}\,\theta)}\right)+\text{Sin}^2\theta\,\dfrac{1}{a(1+\text{Cos}\,\theta)}}{(1+\text{Cos}\,\theta)^2}$$

$$=\frac{(1+\text{Cos}\,\theta)\,\text{Cos}\,\theta+\text{Sin}^2\theta}{a\,(1+\text{Cos}\,\theta)^3}\quad\frac{1+\text{Cos}\,\theta}{a(1+\text{Cos}\,\theta)^3}$$

$$\Rightarrow\left[\frac{d^2y}{dx^2}\right]_{\frac{\pi}{2}}=\frac{1+0}{a(1+0)^3}=\frac{1}{a}.$$

Example 5. If $y=\log\,(x+\sqrt{x^2+a^2})$,

show that $(x^2+a^2)y_2+xy_1=0$.

Solution. $y=\log\,(x+\sqrt{x^2+a^2})$

$$\Rightarrow y_1=\frac{1}{x+\sqrt{x^2+a^2}}\left[1+\frac{2x}{2\sqrt{x^2+a^2}}\right]$$

$$=\frac{1}{\sqrt{x^2+a^2}},$$

squaring both sides we get

$$y_1^2=\frac{1}{x^2+a^2} \text{ or } y_1^2(x^2+a^2)=1.$$

Differentiating this equation w.r.t. x, we get

$y_1^2(x)+(x^2+a^2)\,y_1\,y_2=0$, (It is to be noted that the derivative of y_1^2 is $2y_1\,y_2$).

Or $(x^2+a^2)\,y_2+xy_1=0$. (Since $x^2+a^2\neq0$, $y_1\neq0$.)

Exercise 6.1

1. Find the fourth derivative of $x^{10}+x^9+2x^3-x-1$.
2. Find $f'''(x)$, when $f(x)=\text{Tan}^{-1}x$.
3. Find the value of the fourth derivative of $\frac{1-x}{1+x}$ at $x=0$.
4. Find y^4, when
   (*i*) $y=x^5\,e^x$ at $x=1$
   (*ii*) $y=e^x\ln x$ at $x=1$
   (*iii*) $y=\text{Cos}\,x\,(e^x+e^{-x})$ at $x=0$
   (*iv*) $y=\text{Sin}\,x\ln x$.
   (*v*) $y=\text{Sin}^{-1}x$.
5. If $y=a\,e^{mx}+b\,e^{-mx}$, show that $\frac{d^2y}{dx^2}=m^2y$.
6. If $y=\frac{ax+b}{cx+d}$, show that $2y_1\,y_3=3(y_2)^2$.
7. If $y=e^{\text{Sec}\,x}$ find y_2.
8. If $y=\text{Tan}\,(c\,\text{Tan}^{-1}x)$, show that
   $$(1+x^2)\,y_1=c\,(1+y^2).$$
9. If $y=\text{Sin}\,(m\,\text{Sin}^{-1}x)$, prove that
   $$(1-x^2)y_2=xy_1-m^2y.$$

10. If $f(x)=(\text{Sin}^{-1} x)^2$, prove that
$$(1-x^2) f''(x)-x f'(x)-2=0.$$

11. If $y= l \text{ Cos } x+m \text{ Sin } x$, show that
$$y_2+y=0.$$

12. Find y_2 for the following :
    (*i*) $x=a \text{ Cos } \theta,\ y=a \text{ Sin } \theta$
    (*ii*) $x=at^2,\ y=bt^4$
    (*iii*) $x=a(\text{Cos } \theta+\theta \text{ Sin } \theta),\ y=a(\text{Sin } \theta-\theta \text{ Cos } \theta)$.

13. Find $\frac{d^2y}{dx^2}$ when
    (*i*) $x^2+y^2=a^2$
    (*ii*) $y^2=4ax$
    (*iii*) $ax^2+2h\,xy+by^2=0$
    (*iv*) $x^2+y^2+2gx+2fy+c=0$.

14. If $y=e^{ax}(\text{Sin } bx)$, show that
$$y_2-2ay_1+(a^2+b^2)y=0.$$

15. If $x=\text{Sin } \theta,\ y=\text{Sin } m\theta$, prove that
$$(1-x^2)y_2-xy_1+m^2y=0.$$

ANSWERS

1. $5040\,x^6+3024\,x^5$

2. $\frac{2}{(1+x^2)^2}\left[\frac{4x^2}{(1+x^2}-1\right]$

3. 48.

4. (*i*) $501\,e$
   (*ii*) 0
   (*iii*) -8

(iv) $\frac{-6\,\text{Sin}x}{x^4}+\frac{8\,\text{Cos}\,x}{x^3}+\frac{6\,\text{Sin}\,x}{x^2}-\frac{4\,\text{Cos}\,x}{x}$
$+\text{Sin}\,x\,lnx$

(v) $\frac{9x}{(1-x^2)^{\frac{5}{2}}}+\frac{15x^3}{(1-x^2)^{\frac{7}{2}}}$

7. $e^{\text{Sec}\,x}(\text{Sec}^4x+2\,\text{Sec}^3x-\text{Sec}^2x-\text{Sec}\,x)$

12. (i) $\frac{-a^2}{y^3}$ (ii) $\frac{2b}{a^2}$ (iii) $\frac{1}{a\theta\text{Cos}^3\theta}$

13. (i) $\frac{-a^2}{y^3}$

(ii) $\frac{-4a^2}{y^3}$

(iii) $\frac{2h^3xy+bh^2y^3-2abhxy-ab^2y^2-a^2bx^2+ah^2x^2}{(hx+by)^3}$

(iv) $\frac{(x+g)^2+(y+f)^2}{(y+f)^3}$

6.2. *Calculation of the nth Derivative. (Standard Results).*

6.2.1. *The nth derivative of $(ax+b)^m$.*

Let $y=(ax+b)^m$ So that

$$y_1=ma(ax+b)^{m-1}$$
$$y_2=m(m-1)a\,(ax+b)^{m-2}$$
$$y_3=m(m-1)(m-2)a^3(ax+b)^{m-3}$$

....

....

$$y_n=m\,(m-1)(m-2)\cdots(m-n+1)\;a^n(ax+b)^{m-n}\quad\text{...(1).}$$

If n is a positive integer, y_n can be written as

$$y_n=\frac{m\,!}{(m-n)\,!}\;a^n(ax+b)^m\quad\text{......(2).}$$

Corollory 1. Putting $m=n$ in (2) we get

$$\frac{d^n}{dx^n}(ax+b)^n=\frac{n\,!}{0\,!}a^n(ax+b)^0=n\,!\;a^n.$$

Corollory 2. Putting $a=1$ and $b=0$ in (2) we get

$$\frac{d^n}{dx^n}(x^m)=\frac{m\,!}{(m-n)\,!}\;x^{m-n}.$$

Corollory 3. Putting $m=-1$ in (1), we get

$$\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right)=(-1)(-2)(-3)\cdots\cdots(-n)a^n(ax+b)^{-1-n}$$

$$=\frac{(-1)^n\;n\,!\;a^n}{(ax+b)^{n+1}}.$$

6 2.2. *The nth Derivative of ln (ax+b).*

Let $y=ln(ax+b)$. Thus

$$y_1=\frac{a}{(ax+b)}=a(ax+b)^{-1}$$

$$y_2=a^2(-1)(ax+b)^{-2}$$

$$y_3=a^3(-1)(-2)(ax+b)^{-3}=(-1)^2a^3 2\,!\;(ax+b)^{-3}$$

$$y_4=a^4(-1)(-2)(-3)(ax+b)^{-4}=(-1)^3a^4 3\,!(ax+b)^{-4}$$

....

....

$$y_n=(-1)^{n-1}a^n(n-1)\,!\;(ax+b)^{-n}$$

$$=\frac{(-1)^{n-1}a^n(n-1)\,!}{(ax+b)^n}.$$

Corollory. The *nth* derivative of x^n is

$$\frac{(-1)^{n-1}(n-1)\,!}{x^n}.$$

Example 1. Find the *nth* derivative of $(x+2)^{-3}$.

Solution. $(x+2)^{-3}$ is of the type $(ax+b)^m$, where $a=1$ and $m=-3$.

$$\text{Thus } \frac{d^n}{dx^n}(x+2)^{-3} = (-3)(-4)(-5)\cdots\cdots(-3-n+1)(1)^n(x+2)^{-3-n}$$

$$=(-1)^n.3.4.5\cdots\cdots(n+2)(x+2)^{-n-3}$$

$$=\frac{(-1)^n(x+2)!}{2(x+2)^{n+3}}.$$

Example 2. Find the 25th derivative of $ln\ (4x^2-9)$.

Solution. $ln(4x^2-9) = ln(2x+3)(2x-3)$
$= ln(2x+3) + ln(2x-3)$.

Hence from § 6.2.2 putting $n=25$ and $a=2$. we get

$$_{25}\{ln(4x^2-9)\} = y_{25}\{ln(2x+3)\} + y_{25}\{ln(2x-3).$$

$$=\frac{2^{25}(-1)^{24}24!}{(2x+3)^{25}}+\frac{2^{25}(-1)^{24}24!}{(2x-3)^{25}}$$

$$=2^{25}.24!\left\{\frac{1}{(2x+3)^{25}}+\frac{1}{(2x-3)^{25}}\right\},$$

6.2.3. *The nth derivative of e^{mx}*

Let $y=e^{mx}$

then, $y_1 = me^{mx}$

$y_2 = m^2e^{mx}$

$y_3 = m^3e^{mx}$

……………………

……………………

$y_n = m^ne^{mx}$

Corollory Since $a^x = e^{x\ lna}$, we have

$$y_n(a^x) = y_n(e^{x\ lna}) = (lna)^n\ e^{x lna}$$

$$=(lna)^n a^x.$$

6.2.4 *The nth derivative of Sin $(ax+b)$.*

Let $y = \mathrm{Sin}\,(ax+b)$

then $y_1 = a\,\mathrm{Cos}\,(ax+b) = a\,\mathrm{Sin}\left(\frac{\pi}{2} + \overline{ax+b}\right)$

$$y_2 = a^2\mathrm{Cos}\left(\frac{\pi}{2} + ax + b\right) = a^2\,\mathrm{Sin}\left(\frac{\pi}{2} + \frac{\pi}{2} + \overline{ax+b}\right)$$

$$= a^2\mathrm{Sin}\left(\frac{2\pi}{2} + \overline{ax+b}\right)$$

$$y_3 = a^3\mathrm{Cos}\left(\frac{2\pi}{2} + \overline{ax+b}\right) = a^3\,\mathrm{Sin}\left(\frac{3\pi}{2} + \overline{ax+b}\right)$$

……………………………………………

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$$y_n = a^n\mathrm{Sin}\left(\frac{n\pi}{2} + ax + b\right).$$

Corollary. We can similarly show that the nth derivative of Cos $(ax+b)$ is $a^n\mathrm{Cos}\left(\frac{n\pi}{2} + ax + b\right)$

6.2.5. *The nth derivative of e^{ax} Sin $(bx+c)$.*

Let $y = e^{ax}\mathrm{Sin}\,(bx+c)$

$$y_1 = e^{ax}a\mathrm{Sin}(bx+c) + e^{ax}b\mathrm{Cos}\,(bx+c)$$

$$= e^{ax}\{a\mathrm{Sin}(bx+c) + b\mathrm{Cos}\,(bx+c)\}.$$

Put $a = r\mathrm{Cos}\,\varnothing$ and $b = r\mathrm{Sin}\,\varnothing$, so that

$(a^2+b^2)^{\frac{1}{2}} = r$ (taking positive square root only)

and $\varnothing = \mathrm{Tan}^{-1}\,\dfrac{b}{a}$. Whence

$$y_1 = e^{ax}.\ r\{\mathrm{Sin}(bx+c)\mathrm{Cos}\,\varnothing + \mathrm{Cos}(bx+c)\mathrm{Sin}\,\varnothing\}$$

$$= e^{ax}.r\,\mathrm{Sin}(bx+c+\varnothing).$$

Again $y_2 = r\,e^{ax}\{a\,\mathrm{Sin}(bx+c+\varnothing) + b\mathrm{Cos}(bx+c+\varnothing)\}$

$$= r^2e^{ax}\mathrm{Sin}(bx+c+2\varnothing).$$

Similarly $y_3 = r^3 e^{ax}\,\text{Sin}(bx+c+3\varnothing)$. Continuing we get $y_n = r^n e^{ax}\,\text{Sin}(bx+c+n\varnothing)$.

$$\text{Hence } y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\,\text{Sin}\left(bx+c+n\,\text{Tan}^{-1}\frac{b}{a}\right).$$

Corollory. We can similarly show that the nth derivative of $e^{ax}\text{Cos}\,(bx+c)$ is

$$(a^2+b^2)^{\frac{n}{2}} e^{ax}\,\text{Cos}\left(bx+c+n\,\text{Tan}^{-1}\frac{b}{a}\right)$$

Example 1. Find the nth derivative of $\text{Sin}^2x\text{Cos}^2x$.

Solution. We know that $\text{Cos}\,2x = 1-2\text{Sin}^2x$
$= 2\text{Cos}^2x - 1.$

Thus $\text{Sin}^2x = \frac{1}{2}(1-\text{Cos}\,2x)$ and $\text{Cos}^2x = \frac{1}{2}(1+\text{Cos}2x)$, so that $\text{Sin}^2x\text{Cos}^2x = \frac{1}{4}(1-\text{Cos}^2 2x)$.

Again $\text{Cos}4x = 2\text{Cos}^2 2x - 1$, and so
$\text{Cos}^2 2x = \frac{1}{2}(1+\text{Cos}\,4x)$, whence

$$\text{Sin}^2x\text{Cos}^2x = \tfrac{1}{4}\{1-\tfrac{1}{2}(1+\text{Cos}4x)\}$$
$$= \tfrac{1}{4}-\tfrac{1}{8}+\tfrac{1}{4}\,\text{Cos}4x = \tfrac{1}{8}+\tfrac{1}{4}\,\text{Cos}\,4x.$$

Let $y = \frac{1}{8}+\frac{1}{4}\,\text{Cos}\,4x$. Hence by the corollory of § 6.2.4. we have

$$y_n = \tfrac{1}{4}.\,4^n\,\text{Cos}\left(4x+\frac{n\pi}{2}\right)$$
$$= 4^{n-1}\text{Cos}\left(4x+\frac{n\pi}{2}\right).$$

Example 2. Find the 16*th* derivative of $e^{\sqrt{3}\,x}\,\text{Sin}^3x$.

Solution. We know that $\text{Sin}\,3x = 3\text{Sin}x - 4\text{Sin}^3x$.

So that $\text{Sin}^3x = \frac{1}{4}(3\text{Sin}\,x - \text{Sin}3x) = \frac{3}{4}\,\text{Sin}\,x - \frac{1}{4}\,\text{Sin}\,3x$.

Let $y = e^{\sqrt{3x}}\,\text{Sin}^3x$, thus $y = \frac{3}{4}e^{\sqrt{3x}}\,\text{Sin}\,x$
$- \frac{1}{4}e^{\sqrt{3}\,x}\,\text{Sin}\,3x$

Hence by §6.2.5 we have

$$y_{16}=\tfrac{3}{4}(3+1)^{\frac{16}{2}}\,e^{\sqrt{3}\,x}\,\text{Sin}\left(x+16\,\text{Tan}^{-1}\frac{1}{\sqrt{3}}\right)-\tfrac{1}{4}(3+9)^{\frac{16}{2}}$$

$$-\tfrac{1}{4}(3+9)^{\frac{16}{2}}e^{\sqrt{3}\,x}\left(\text{Sin}\;3x+16\text{Tan}^{-1}\frac{3}{\sqrt{3}}\right)$$

$$=e^{\sqrt{3}\,x}\left\{\tfrac{3}{4}.\;4^{8}\text{Sin}\;\left(x+16\frac{\pi}{6}\right)-\tfrac{1}{4}.\;12^{8}\;\text{Sin}\;\left(3x+16\frac{\pi}{3}\right)\right\}$$

$$=e^{\sqrt{3}\,x}\left\{3.\;4^{7}\text{Sin}\left(x+\frac{8\pi}{3}\right)-\tfrac{1}{4}.\;12^{8}\text{Sin}\left(3x+\frac{16\pi}{3}\right)\right\}.$$

Exercise 6.2.

1. Find the nth derivatives of the following :

(*i*) $x^{\frac{1}{7}}$.

(*ii*) x^{-5}.

(*iii*) $(\sqrt{x})^{-\frac{2}{3}}$.

(*iv*) $\dfrac{1}{x^{p}}$.

(*v*) $\dfrac{1}{(ax+b)^{p}}$

(*vi*) $\dfrac{1}{2x-4}$

(*vii*) $^{\sqrt{2}\,x}$

(*viii*) 2^{3x}.

(*ix*) $(\frac{1}{2})^{5x}$.

2. Find the 20*th* derivative of $\dfrac{x}{x-2}$

3. Find the *n*th derivatives of :

   (*i*) $\ln (x-a)(x-b)(x-c)$.

   (*ii*) $\ln (x^2-9)$.

   (*iii*) $\ln \left(\dfrac{3x+7}{2x-9}\right)$.

   (*iv*) $\ln (x^2+2x)$.

4. Find the 15*th* derivative of $\dfrac{1+x}{x^2+5x+6}$.

(**Hint**. Decompose the given function into partial fractions and proceed.)

5. Find the *n*th derivatives of:

   (*i*) $\dfrac{x}{9x^2-1}$.

   (*ii*) $\dfrac{x^3+x^2+x+1}{(x+1)(x+2)}$.

   (*iii*) $\dfrac{1}{(x^2-4)(x^2-9)}$.

   (*v*) $\dfrac{x^2}{(x-1)(x-2)(x-3)}$.

6. Find the *n*th derivatives of :

   (*i*) $\text{Sin}^2 x$. (*ii*) $\text{Cos}^3 x$.

   (*iii*) $\text{Sin}x \,\text{Cos}2x$. (*iv*) $\text{Sin}^4 x$.

   (*v*) $\text{Cos}^4 x$. (*vi*) $\text{Sin}\, x \text{Cos}^3 x$.

7. Find the *n*th derivatives of :

   (*i*) $e^x \text{Cos}^2 x$.

   (*ii*) $e^{-2x} \text{Sin}\, 3x \text{Cos} 4x$.

   (*iii*) $e^{\sqrt{5}\, x}\, \text{Cos}\, 2x\, \text{Cos}\, 5x$.

   (*iv*) $e^{-\frac{1}{2}x}\, \text{Sin}^2 (4x-2)$.

ANSWERS

1. (i) $(-1)^{n-1} \frac{1}{7^n} 6.13 \ldots\ldots (7n-6) x^{\frac{1}{7}-n}$

(ii) $(-1)^n 5.6.7 \ldots\ldots (n+4) x^{5-n}$

(iii) $(-1)^n \frac{1}{3^n} 1.4.7 \ldots\ldots (3n-2) x^{-\frac{1}{3}-n}$

(iv) $\frac{(-1)^n (p+n-1)!}{(p-1)!\, x^{p+n}}$ (p is an integer)

(v) $\frac{a^n (-1)^n (p+n-1)!}{(p-1)!\, (ax+b)^{p+n}}$ (p is an integer)

(vi) $\frac{2^n (-1)^n\, n!}{(2x-4)^{n+1}}$

(vii) $(2)^{\frac{n}{2}} e^{\sqrt{2}\, x}$

(viii) $3^n (\ln 2)^n 2^{3x}$

(ix) $5^n (\ln \frac{1}{2})^n (\frac{1}{2})^{5x}$

2. $\frac{2(20!)}{(x-2)^{21}}$

3. (i) $(-1)^{n-1}(n-1)! \left\{ \frac{1}{(x-a)^n} + \frac{1}{(x-b)^n} + \frac{1}{(x-c)^n} \right\}$

(ii) $(-1)^{n-1}(n-1)! \left\{ \frac{1}{(x-3)^n} + \frac{1}{(x+3)^n} \right\}$

(iii) $(-1)^{n-1}(n-1)! \left\{ \frac{3^n}{(3x+7)^n} - \frac{2^n}{(2x-9)^n} \right\}$

(iv) $(-1)^{n-1}(n-1)! \left\{ \frac{1}{x^n} + \frac{1}{(x+2)^n} \right\}$

$15! \left\{ \frac{2}{(x+3)^{16}} - \frac{1}{(x+2)^{16}} \right\}$

5. (i) $\frac{1}{8}(-1)^n 3^n n!\left\{\frac{1}{(3x-1)^{n+1}}+\frac{1}{(3x+1)^{n+1}}\right\}$

(ii) $\frac{5(-1)^n . n!}{(x+2)^{n+1}}$.

(iii) $(-1)^n n!\left\{\frac{\frac{1}{20}}{(x+2)^{n+1}}-\frac{\frac{1}{20}}{(x-2)^{n+1}}\right.$

$\left.+\frac{\frac{1}{30}}{(x-3)^{n+1}}-\frac{\frac{1}{30}}{(x+3)^{n+1}}\right\}$

(iv) $(-1)^n n!\left\{\frac{\frac{1}{2}}{(x-1)^{n+1}}-\frac{4}{(x-2)^{n+1}}+\frac{\frac{9}{2}}{(x-3)^{n+1}}\right\}$

6. (i) $2^{n-1}\,\mathrm{Sin}\left\{2x+\frac{(n-1)\pi}{2}\right\}$

(ii) $\frac{1}{4}\left\{3^n\mathrm{Cos}\left(3x+\frac{n\pi}{2}\right)+3\,\mathrm{Cos}\left(x+\frac{n\pi}{2}\right)\right\}$

(iii) $\frac{1}{2}\left\{3^n\mathrm{Sin}\left(3x+\frac{n\pi}{2}\right)-\mathrm{Sin}\left(x+\frac{n\pi}{2}\right\}\right.$

(iv) $\frac{1}{8}\left\{4^n\mathrm{Cos}\left(4x+\frac{n\pi}{2}\right)-4.\,2^n\mathrm{Cos}\left(2x+\frac{n\pi}{2}\right)\right\}$

(v) $\left\{4^{n-1}\mathrm{Cos}\left(4x+\frac{n\pi}{2}\right)+2^{n-1}\,\mathrm{Cos}\left(2x+\frac{n\pi}{2}\right)\right\}$

(vi) $\frac{1}{2}\left\{4^{n-1}\mathrm{Sin}\left(4x+\frac{n\pi}{2}\right)+2^{n-1}\mathrm{Sin}\left(2x+\frac{n\pi}{2}\right)\right\}$

7. (i) $\frac{1}{2}e^x+\frac{1}{2}(5)^{\frac{n}{2}}\,e^x\,\mathrm{Cos}(2x+n\mathrm{Tan}^{-1}2)$

(ii) $\frac{1}{2}\{(53)^{\frac{n}{2}}e^{-2x}\mathrm{Sin}\,[7x+n\mathrm{Tan}^{-1}(-\frac{7}{2})]$

$-(5)^{\frac{n}{2}}\,\mathrm{Sin}[x+n\mathrm{Tan}^{-1}(-\frac{1}{2})]\}$

(iii) $\frac{1}{2}\left\{e^{\sqrt{5}x}\,(54)^{\frac{n}{2}}\,\mathrm{Cos}\left(7x+n\mathrm{Tan}^{-1}\frac{7}{\sqrt{5}}\right)\right.$

$\left.+(14)^{\frac{n}{2}}\mathrm{Cos}\left(3x+n\mathrm{Tan}^{-1}\frac{3}{\sqrt{5}}\right)\right\}$

(iv) $\frac{1}{2}(-\frac{1}{3})^n e^{-\frac{1}{3}x} - \frac{1}{2}(\frac{1}{9}+64)^{\frac{n}{2}} e^{-\frac{1}{3}x}$
$\cdot \text{Cos}(8x-4+n\text{Tan}^{-1}(-24))\}$

.3. *Expansion of Functions* :

We are now taking up the problem of expanding a given function $f(x)$ in an infinite series of ascending integral powers of x. In other words we want to obtain an infinite series which is convergent and whose sum is equal to the function $f(x)$ We will state here only two theorems, namely 'Maclaurin's Theorem, and 'Taylor's Theorem' Their proofs are beyond the scope of the present book.

6.3.1. *Maclaurin's Theorem.*

If $f(x)$ is a function that can be expanded in an infinite series of positive incgral ascending powers of x then,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots\dots$$

It is, however pointed out that Maclaurin's Theorem cannot be used to expand a function $f(x)$ if any one of $f(0)$ $f'(0), f''(0),\dots\dots$does not exist or is infinite. Also the expansion is invalid if the series,

$$f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots\dots$$

does not converge, though the students are not expected to test their convergence at this stage.

6.3.2. *Taylor's Theorem.*

If $f(x+h)$ is a function that can be expanded in an infinite series of positive integral ascending powers of h, then,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\dots\dots\dots\quad(1)$$

It is to be noted that Taylor's Theorem fails to expand $f(x+h)$ if any one of $f(x), f'(x), f''(x),\dots\dots\dots$ does not exist or is infinite. Also the expansion is invalid if the series obtained are divergent.

Corollory We can deduce Maclaurin's Theorem from Taylor's Theorem by first putting $x=0$ and then replacing h by x. Thus by puttihg $x=0$ in (1), we get

$$f(h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(0)+\frac{h^3}{3!}f'''(0)+\dots\dots\dots\dots$$

Now changing h to x, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots\dots$$

Example 1. Expand Cos x by Maclaurin's Theorem and deduce the value of Cos 18°.

Solution. Let $f(x)=\text{Cos}x$, thus $f(0)=\text{Cos}0=1$.

$$\begin{aligned} f'(x)&=-\text{Sin }x, & f'(0)&=0.\\ f''(x)&=-\text{Cos }x, & f''(0)&=-1.\\ f'''(x)&=\text{Sin }x, & f'''(0)&=0.\\ f^4(x)&=\text{Cos }x, & f^4(0)&=1.\end{aligned}$$

Now by Maclaurin's Theorem

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots\dots$$

Hence $\text{Cos}x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\dots\dots\dots$

To obtain the value of Cos 18° we proceed as follows :

$$18^\circ=\frac{18\pi}{180}\text{ radians}=\frac{\pi^c}{10}=\frac{3.14}{10}=.314$$

$$\therefore \text{Cos }18=\text{Cos}\cdot 314=1-\frac{(\cdot 314)^2}{2\,!}+\frac{(\cdot 314)^4}{4\,!}-\dots\dots\dots\dots$$

$$=\cdot 9511 \text{ approx.}$$

Example 2. Expand 2^{x+y} in powers of y by Taylor's Theorem and deduce the expansion of 2^y.

Solution. Let $f(x+y) = 2^{x+y}$. Putting $y=0$ we get

$$f(x) = 2^x.$$
$$f'(x) = 2^x ln2.$$
$$f''(x) = 2^x (ln2)^2.$$
$$f'''(x) = 2^x (ln2)^3.$$

……………………

……………………

Now by Taylor's Theorem

$$f(x+y) = f(x) + y f'(x) + \frac{y^2}{2!} f''(x) + \cdots\cdots\cdots$$

Thus $$2^{x+y} = 2^x + y 2^x (ln2) + \frac{y^2}{2!} 2^x (ln2)^2 + \cdots\cdots\cdots$$

$$= 2^x \left\{ 1 + y(ln2) + \frac{y^2}{2!} (ln2)^2 + \cdots\cdots\cdots \right\}$$

Putting $x=0$, we get

$$2^y = 1 + y\,(ln2) + \frac{y^2}{2!} (ln2)^2 + \cdots\cdots\cdots\cdots\cdots$$

Example 3. Expand $ln\ (1+x)$ by Maclaurin' Theorem.

Solution. Let $f(x) = ln\ (1+x)$, so that

$$f(0) = ln 1 = 0$$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}, \quad \therefore \quad f'(0) = 1$$
$$f''(x) = -(1+x)^{-2}, \qquad f''(0) = -1 = -1!$$
$$f'''(x) = (-1)(-2)(1+x)^{-3}, \qquad \therefore\ f'''(0) = 2\,!$$
$$f^4(x) = (-')(-2)(-3)(1+x)^{-4} \quad \therefore\ f^4(0) = -\cdot 3\,!$$

……………………………………

……………………………………

Now by Maclaurin's Theorem

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\dots\dots\dots\dots\dots$$

Hence $$ln(1+x)=x+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2\,!)$$

$$+\frac{x^4}{4!}(-3\,!)+\dots\dots\dots$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\dots\dots$$

Exercise 6.3.

1. Apply Maclaurin's Theorem to prove that :

(*i*) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\dots\dots\dots\dots$

Deduce the value of *e*.

(*ii*) $\text{Sin}x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\dots\dots\dots$

(*iii*) $ln(1-x^2)=-x^2-\frac{x^4}{2}-\frac{x^6}{3}-\dots-\frac{x^{2n}}{n}-\dots\dots\dots$

(*iv*) $\frac{e^x}{1+e^x}=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+\dots\dots\dots$

(*v*) $e^{\text{Sin}x}=1+x+\frac{x^2}{2}-\frac{x^4}{8}+\dots\dots\dots$

(*vi*) $\text{Sin}^{-1}x=x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\dots\dots\dots$

Deduce the value of π.

2. Expand $\text{Tan}\left(\frac{\pi}{4}+x\right)$ to 5 terms and hence calculate the value of Tan 46°.

3. Show that $ln\,(1+x+x^2)=x+\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}+\cdots$

Hint. $ln(1+x+x^2)=ln\left\{\frac{(1-x)(1+x+x^2)}{(1-x)}\right\}$

$$=ln\left(\frac{1-x^3}{1-x}\right)=ln(1-x^3)-ln(1-x).$$

Deduce the value of $\log_{10}111$, when $\log_{10}e=\cdot434$.

4. Show that $ln(1-x+x^2)$

$$=-x+\frac{x^2}{2}+\frac{2x^3}{3}+\frac{x^4}{4}-\cdots\cdots\cdots$$

5. Expand Sin $(x+y)$ in powers of y and
   (*i*) Deduce the expansion of Sin$x$ and Cos$x$
   (*ii*) Verify that Sin $(x+y)$ = Sin$x$ Cos$y$ + Cos$x$Sin$y$.

6. Show that $\text{Sin}(x+h)=\text{Sin}x+h\text{Cos}x-$

$$\frac{h^2}{2!}\text{Sin}x-\frac{h^3}{3!}\text{Cos}x+\cdots\cdots$$

Deduce the value of Sin61°.

7. Prove that $\text{Tan}\,(x+h)=\text{Tan}\,x+h\,\text{Sec}^2x$

$$+h^2\,\text{Sec}^2x\,\text{Tan}\,x+\frac{h^2}{3}\text{Sec}^2x\,(1+3\,\text{Tan}^2x)+\cdots\cdots$$

8. Show that $ln\,\text{Sin}\,(x+h)=ln\,\text{Sin}\,x+h\,\text{Cot}\,x$

$$-\frac{h^2}{2}\text{Cosec}^2x+\frac{h^3}{3}\text{Cot}\,x\,\text{Cosec}^2x+\cdots\cdots$$

ANSWERS

1. (*i*) 2·718 approximately.
2. $1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\cdots\cdots\cdots$; 1·03553.
3. 2.0453 approximatelw.
6. ·8746 approximately.

CHAPTER VII

FUNDAMENTALS OF PLANE ANALYTIC GEOMETRY AND STRAIGHT LINE

7.1. *Locus of an Equation.* The solution set of an equation in two unknowns x and y is an infinite set. This set is called the locus of an equation It consists of all points in the coordinate plane whose coordinates (x, y) satisfy the equation. A geometric representation of the locus (*i.e.* the actual drawing) is called the graph of the equation.

We shall use the term "locus of an equation" and "solution set of an equation" interchangeably.

The sketch of the exact graph of an equation would involve the plotting of an infinite number of points. As this is obviously impracticable, we usually select enough points to exhibit the general nature of the locus, plot these points, and approximate the remaining points by drawing a "curve" through the plotted points.

A systematic method of choosing points on the locus is to solve the given equation for one of the unknowns in terms of the other, and then in the resulting relation assign values to the latter in order to determine the corresponding values of the former. The values of the ordered pairs (x, y) so obtained are tabulated, points plotted and the graph drawn.

In sketching the graph of a curve. whose equation is given, the concepts of *intercepts* and *symmetry* play a very vital role. We therefore introduce these concepts.

7.2. *Intercepts and Symmetry.*

Definition. **The x-intercept of a curve is the x-coordinate of the point of intersection of the curve with the x-axis, the y-intercept is defined similarly.**

A curve may have more than one x-intercepts or y-intercepts. The rule for finding them is as follows:

To find the x-intercept put $y=0$ and solve the resulting equation for x. Similarly to find the y-intercept, put $x=0$ and solve for y.

Definitions. Two points A and A′ are *symmetrically* located with respect to a line* l if and only if l is the perpendicular bisector of the segment AA′. A locus is *Symmetric with respect to a line l* if and only if the entire locus consists of pairs of points, symmetrically located with respect to l.

Two points A and A are *symmetrically located with respect to a point B* if and only if B is the mid point of the segment AA′. A locus is symmetric with respect to a point B if and only if the entire locus consists of pairs of points symmetrically located with respect to B.

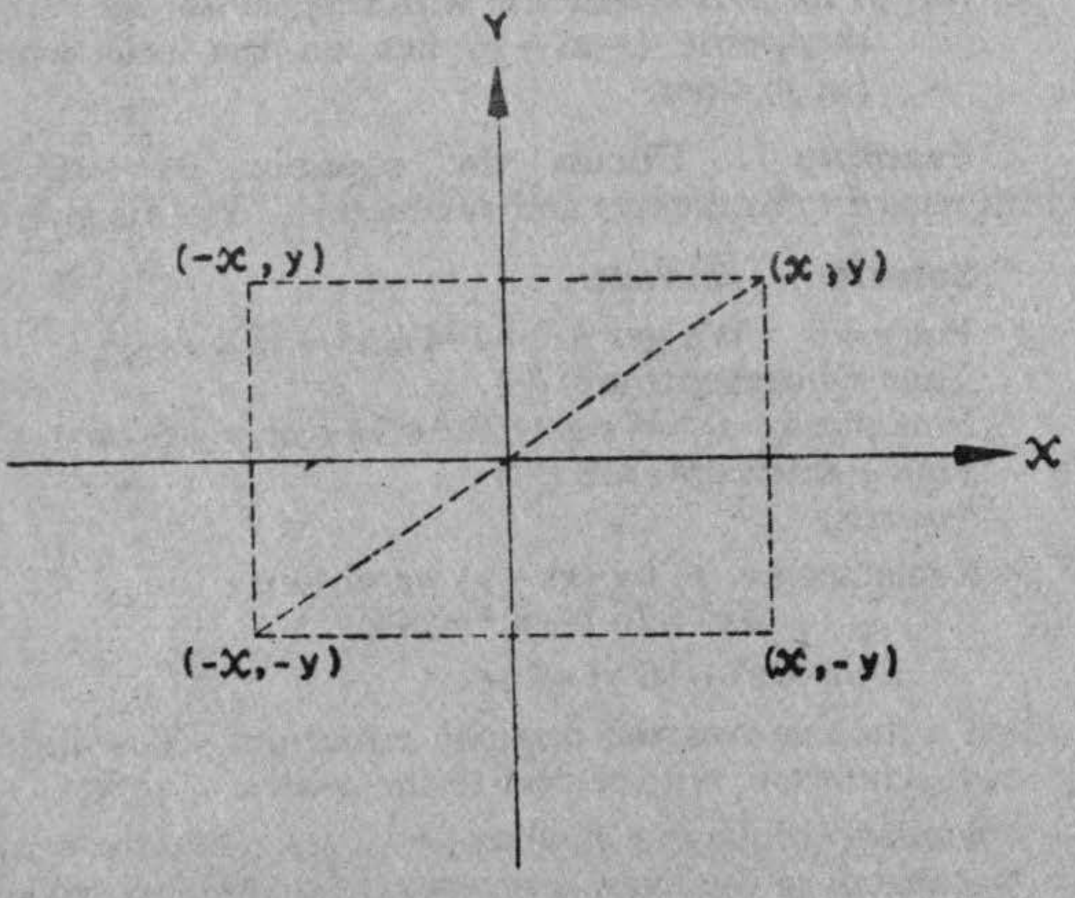

Fig. 7.1

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* By a " line " we mean a straight line.

From Fig. 7.1. we see immediately that two points are symmetrically located with respect to the x-axis if the coordinates of the points are of the form (x, y) and $(x, -y)$. Likewise, they are symmetrically located with respect to the y-axis if their coordinates are of the form (x, y) and $-x, y)$. Finally, two points are symmetrically located with respect to the origin if their coordinates are of the form (x, y) and $(-x, -y)$. Therefore we obtain the following tests for symmetry :

(*i*) A locus is symmetric with respect to the x-axis if the point $(x, -y)$ lies on the locus whenever (x, y) does.

(*ii*) A locus is symmetric with respect to the y-axis if the point $(-x, y)$ lies on the locus whenever (x, y) does.

(*iii*) A locus is symmetric with respect to the origin if the point $(-x, -y)$ lies on the locus whenever (x, y) does.

Example 1. Discuss the equation $9x^2+16y^2=144$ with regard to intercepts and symmetry. Plot its graph.

Solution. ***Intercepts.***

Put $y=0$. We get $9x^2=144 \Rightarrow x^2=16 \Rightarrow x=\pm 4$.
Thus x-intercepts are ± 4.
Now put $x=0$, We get $16y^2=144$ or $y^2=9$ or $y=\pm 3$.
Thus y-intercepts are $\pm$ 3.
Symmetry.
Replacing (x, y) by $(x, -y)$ we obtain

$$9x^2+16(-y)^2=144$$

$$\text{or } 9x^2+16y^2=144$$

which is the same as the original equation. Therefore the locus is symmetric with respect to the x-axis.

Similarly replacing (x, y) by $(-x, y)$ we find the equation is unchanged, so the locus is symmetric with respect to y-axis.

Again replacing (x, y) by $(-x, -y)$, the equation does not change. Hence the locus is symmetric with respect to the origin

Graph.

Solving for y in terms of x, we obtain

$$y = \pm \tfrac{3}{4}\sqrt{16 - x^2}$$

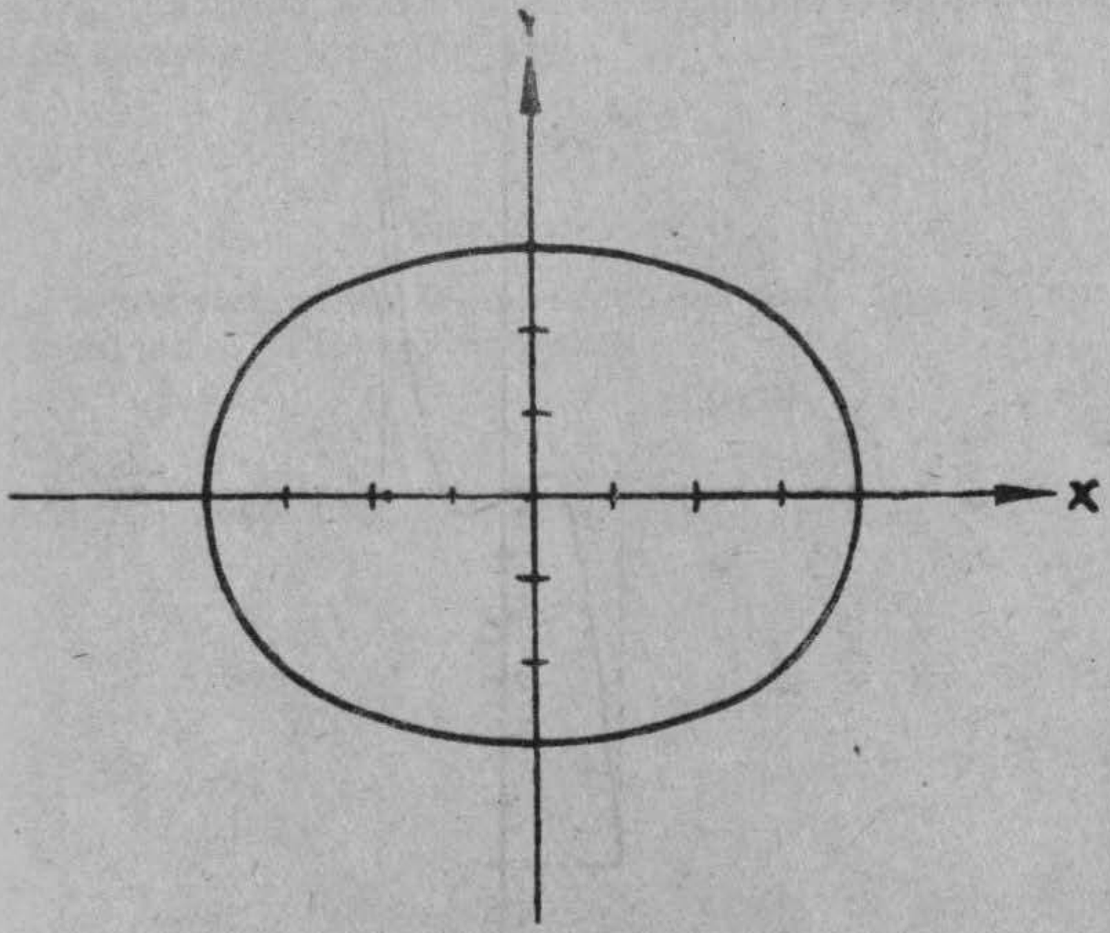


Figure 7.2.

Plotting the points (x, y) we obtain the graph of Figure 7.2. In this case we need to plot the curve in only one quadrant and complete it by taking into consideration its symmetry w.r.t both the axes.

Example 2. Examine the equation $y = x^5 - x$ for intercepts and symmetry. Plot its its graph.

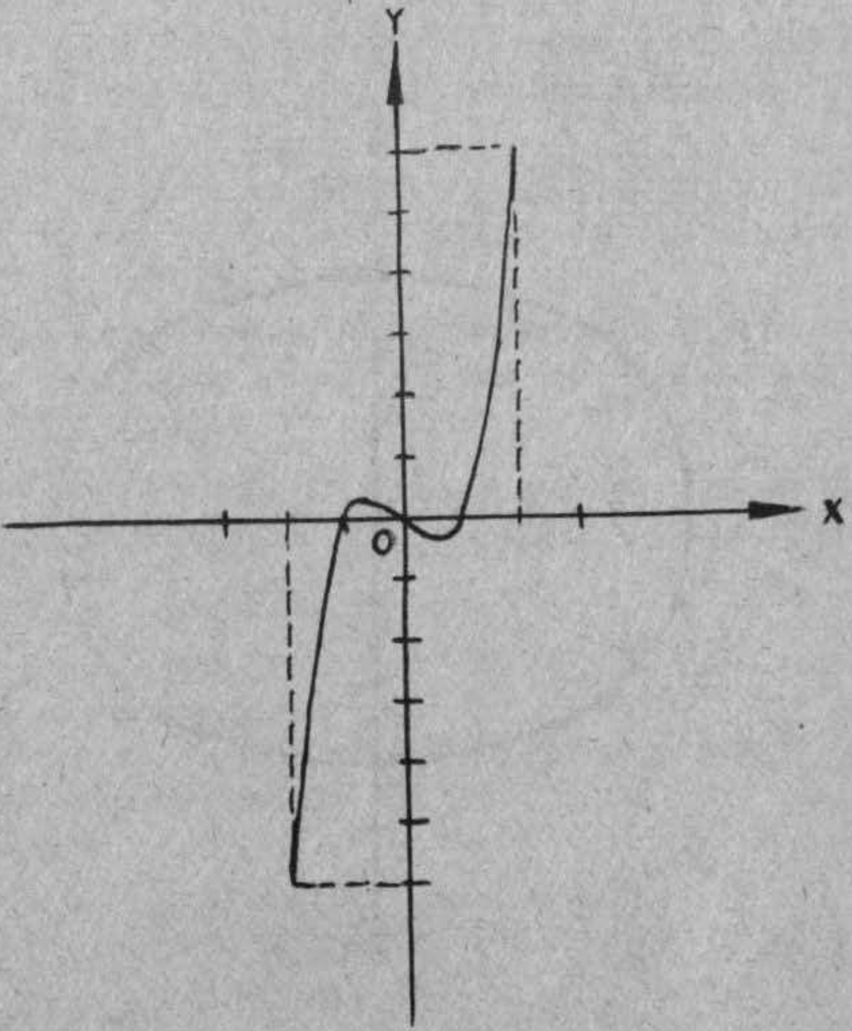


FIG.7.3

Solution. Putting $y=0$, we get $x^3-x=0 \Rightarrow x(x^2-1)=0$. Thus $0, \pm 1$ are x-intercepts.

Putting $x=0$, we get $y=0$, so the y-intercept is zero. We find the locus is symmetric w.r.t the origin and there is no symmetry w.r.t x-axis or y-axis.

Plotting these points (x, y) we obtain the graph of Fig 7.3. In this case we need draw the curve in only two adjacent quadrants, and complete it by taking into consideration its symmetry w.r.t the origin.

Exercise 7.1

Discuss each of the following equations w.r.t intercepts and symmetry. Plot their graphs.

1. $y=4-x^2$
2. $y^2=8x$
3. $y^2=2-x$
4. $x^2+y^2=25$
5. $2x^2+2y^2=32$
6. $2x^2+3y^2=18$
7. $\frac{x^2}{25}+\frac{y^2}{16}=1$
8. $2x^2-3y^2=6$
9. $\frac{x^2}{9}-\frac{y^2}{16}=1$
10. $xy=4$
11. $y=x-x^3$
12. $x=y-y^3$
13. $x^2-y^2=9$
14. $x=\frac{1}{4}y^2$.

7.3. *Intersection of Loci.* Since a point of intersection of two loci lies on each of them, its coordinates must satisfy both the equations. Accordingly the points of intersection may be found by solving the two equations simultaneously.

Example 1. Find the points of intersection of the loci of the equations

$$x+2y=4 \text{ and } y^2=x+4.$$

Plot the graphs of the two loci w.r.t the same axes.

Solution. $x+2y=4$ ...... (*i*)

$y^2=x+4$ ...... (*ii*)

from (*i*) we have $x=4-y$ ......(*iii*)

Substituting the value of x from (*iii*) in (*ii*) we get

$$y^2=8-2y \Rightarrow y^2+2y-8=0 \Rightarrow (y+4)\,(y-2)=0$$

$$\text{or} \quad y=-4,\ 2.$$

From (*iii*) for $y=-4$, we get $x=12$ and for $y=2$, $x=0$. Thus the two points of intersection are $(12, -4)$ and $(0, 2)$.

Graphs of the loci are drawn in figure 7.4.

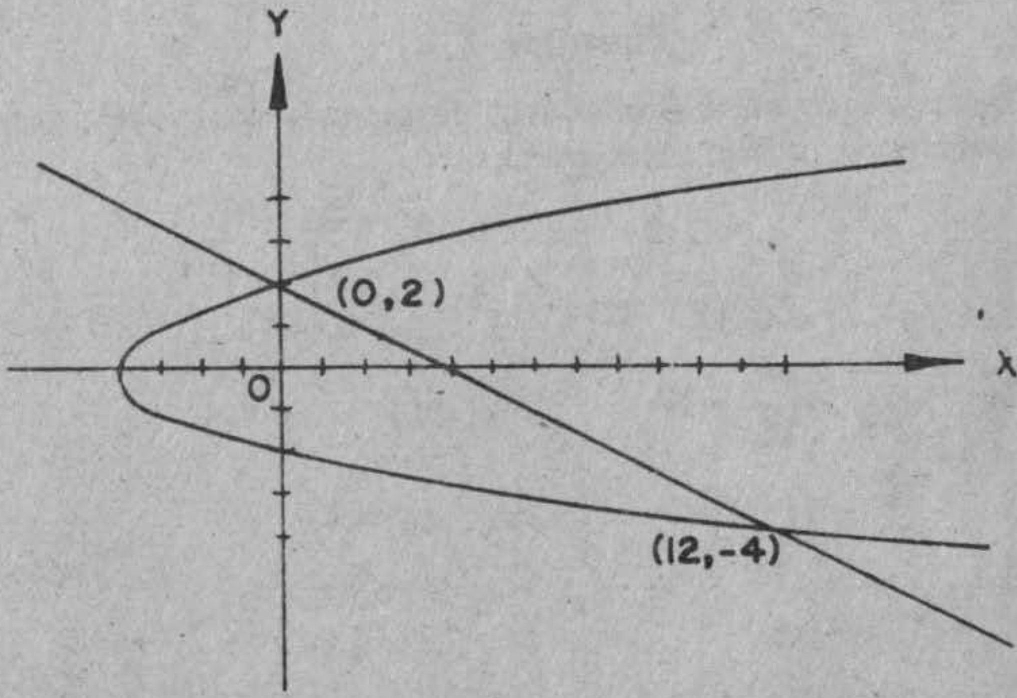


Fig.7.4

Example 2. Find the points of intersection of the curves $2x^2-3y^2=5$ and $4x^2+9y^2=25$.

Illustrate graphically.

Solution. Solving $2x^2-3y^2$ and $4x^2+9y^2=25$ simultaneously we get $(2, 1)$, $(2, -1)$, $(\ 2, 1)$ and $(-2, -1)$ as the four points of intersection. The graphs of the loci are given in Figure 7.5.

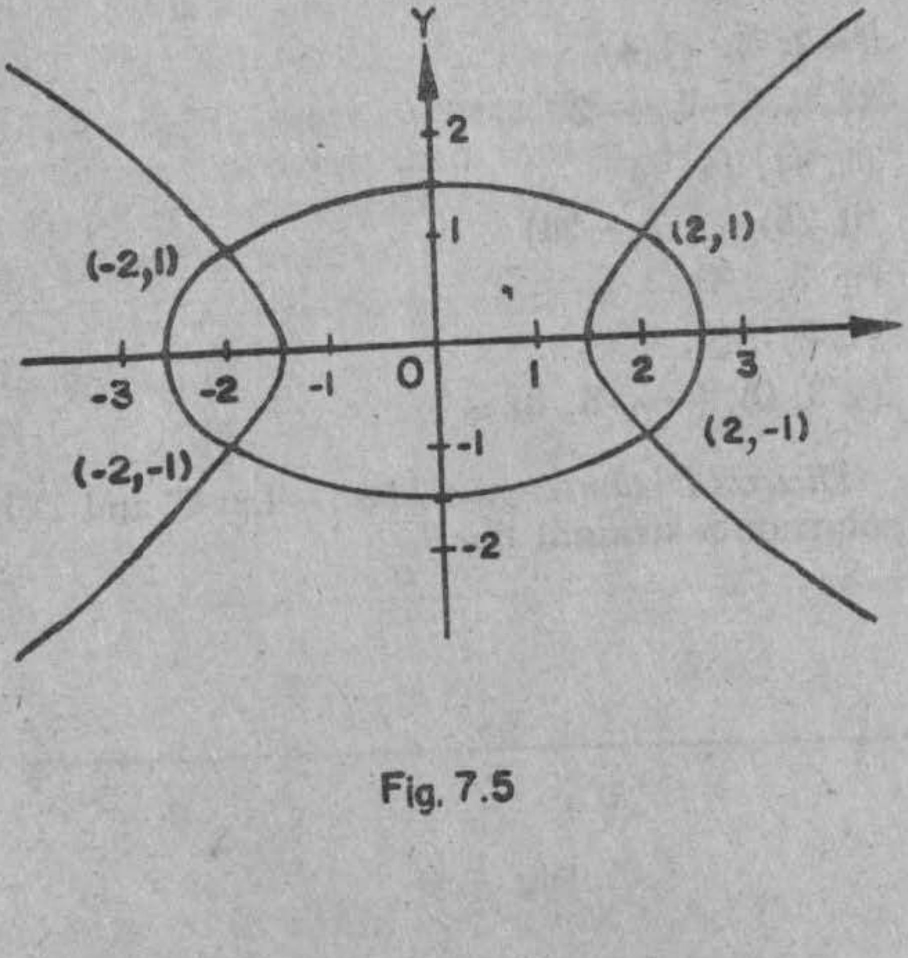

Y
2
1
-1
-2
-3
-2
-1
0
1
2
3
(-2,1)
(2,1)
(-2,-1)
(2,-1)

Fig. 7.5

Exercise 7.2

Find all the points of intersection of the following pairs of loci. Illustrate graphically on the same graph :—

1. $x+y=5$, $y=x^2+3$
2. $x-y=1$, $y=x^2-3$
3. $x+y=5$, $xy=4$

4. $7x-y+25=0, \quad x^2+y^2=625$
5. $x^2+y^2=25, \quad x^2+4y^2=73$
6. $3x+2y=7, \quad 3x^2+8y^2=35$
7. $3x^2+y^2=9, \quad x^2-y^2=3$

ANSWERS

1. $(-2, 7), (1,4)$
2. $(2,1), (-1, -2)$
3. $(1, 4), (4, 1)$
4. $(0, 25), (-7, -24)$
5. $(\pm 3, \pm 4)$
6. $(1, 2), (3, -1)$
7. $(\sqrt{3}, 0), (-\sqrt{3}, 0)$.

7.4. *Directed Distances on a Line* :—Let C and D be two distinct points on a straight line *l*.

Fig. 7.6

We may move along the line from C to D or from D to C. Thus there are two possible directions on *l* and we choose one of them and indicate it by an arrow head on *l*. In figure 7.6 the chosen direction is from D to C. Although the choice of direction is quite arbitrary, when it is once made we consider it to be fixed, and we say that *l* is the directed line having the direction of the arrow head. If A and B are any two points on l, we denote the distance between them by $|AB|$. The directed distance $\overline{AB}$ from A to B is defined to be $|AB|$ or $-|AB|$ according as a particle moving along the line from A to B goes in the direction of the arrow head or in the opposite direction.

Thus in figure 7.6

$\overline{AB} = |AB|$, $\overline{CD} = -|CD|$, $\overline{BA} = -|BA|$ and

$\overline{DC} = |DC|$.

The directed distance between two points A and B differs from the distance between them in that the directed distance may be a positive or a negative number where as the distance is always a positive number.

If A, B, C are any three points on a line l,

$\overline{AB} + \overline{BC} = \overline{AC}$

$\overline{BC} = \overline{AC} - \overline{AB}$.

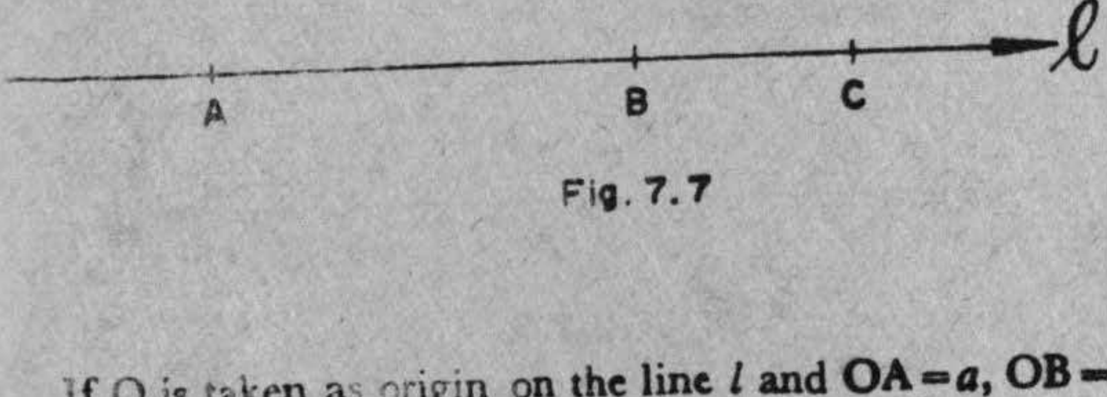


Fig. 7.7

If O is taken as origin on the line l and $OA = a$, $OB = b$ then a and b may be called as the coordinates of A and B respectively on the line *w.r.t.* O as origin.

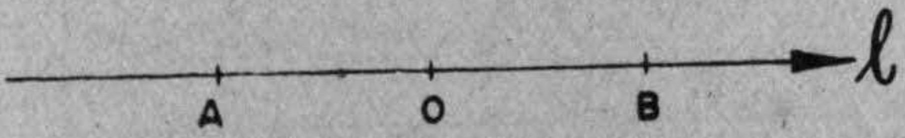


Fig. 7.8

We have $\overline{AB} = \overline{OB} - \overline{OA} = b - a$

$$\overline{BA} = -\overline{AB} = -(b-a) = a-b$$

$$\therefore |AB| = |a-b| = |b-a|.$$

We already know from Modern Algebra and Trigonometry that the distance between two points A (x_1, y_1) and B (x_2, y_2) in a plane is

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

which is known as the distance Formula.

7.5. *Division of a line segment*in a given ratio.*

Theorem 7.5.1 :—If A and B are points of a line and if their coordinates are a and b then the mid point of the segment AB has coordinates

$$\frac{a+b}{2}.$$

Proof : Let c be the coordinate of the middle point C of AB. Then.

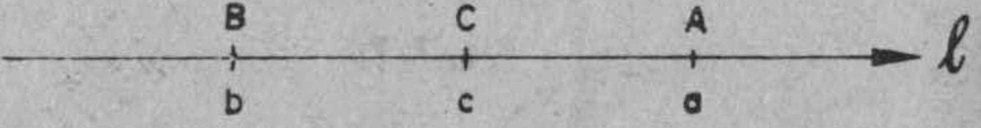


Fig. 7.9

$$\overline{AC} = \overline{CB}$$

$$\therefore \quad c-a = b-c$$

$$i.e. \; c = \frac{a+b}{2}.$$

---

*A line segment' or simply a 'segment' AB is the portion of a line between the points A and B. Alternatively if A and B are points having co-ordinates a and b, on a line l, then segment

$$\overline{AB} = \{x \mid a \leq x \leq b\}$$

Theorem 7.5.2. If A, B and C are three distinct points on a line, with A and B having coordinates a and b respectively, and if

$$\frac{\overline{AC}}{CB} = r,$$

then the point C has coordinate c given by

$$c = \frac{a+rb}{1+r}.$$

Proof : Since $\frac{\overline{AC}}{\overline{CB}} = r,$

$$\frac{c-a}{b-c} = r$$

or $\quad c-a = r\,(b-c)$

or $\quad c+rc = a+rb$

or $\quad (1+r)\,c = a+rb\,.$

Thus $\quad c = \frac{a+rb}{1+r}.$

Theorem 7.5.3 If $P_1\,(x_1, y_1)$, $P_2\,(x_2, y_2)$ and $P\,(x, y)$ are three distinct points on a line, and if

$$\frac{\overline{P_1P}}{\overline{PP_2}} = r$$

then the point P has coordinates

$$\left(\frac{x_1+rx_2}{1+r}.\ \frac{y_1+ry_2}{1+r}\right)$$

w.r.t. a system of coordinate axes.

Proof: Suppose the given line is not parallel to either coordinate axis.

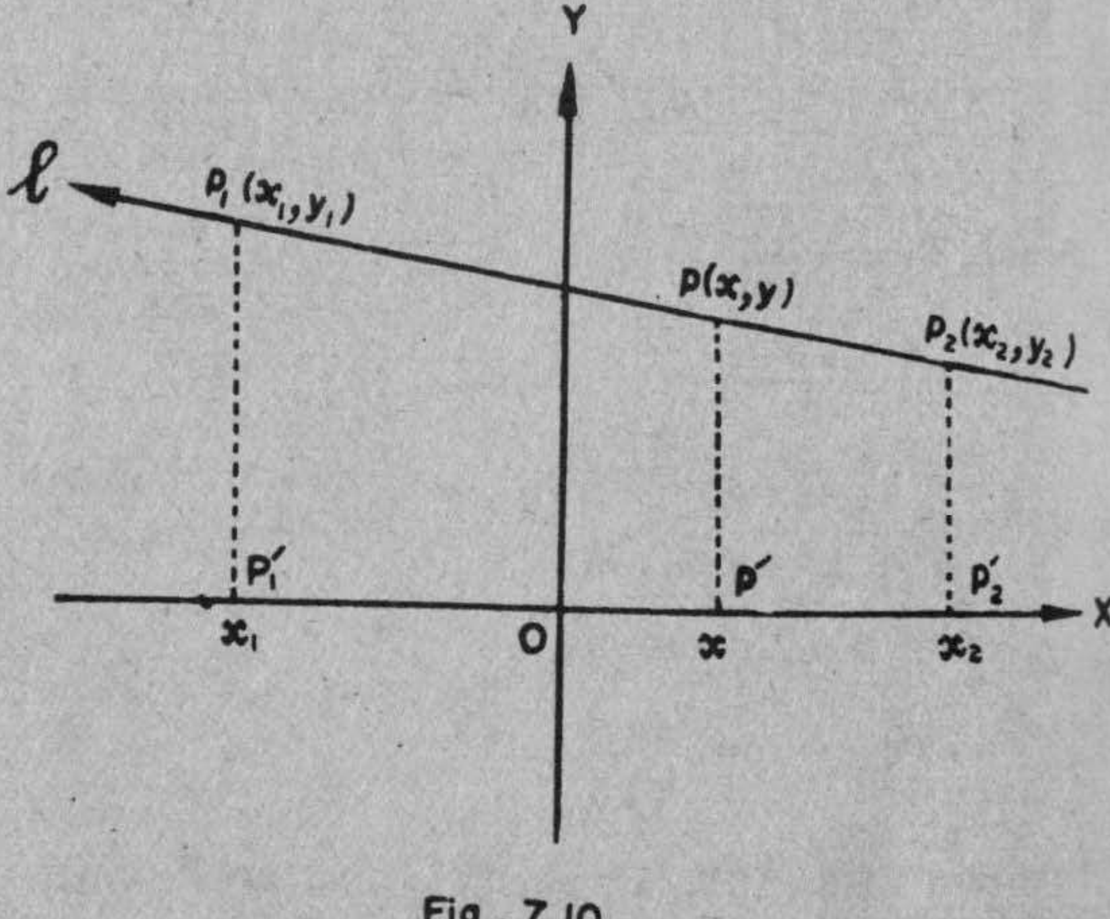


Fig . 7.10

Let P_1', P_2' and P' be the respective projections[*] of P_1 and P_2 and P on the x-axis. Thus P_1', P_2' and P' are points on a coordinate axis (the x-axis) with coordinates x_1, x_2 and x respectively. Now since the line through P_1 and P_1', the line through P_2 and P' and the line through P and P' are all parallel, and these parallel lines are cut by two transversals, namely l and the x-axis, it follows from elementary geometry that

---

[*] Projection of a point *P* on a line *l* is the foot of the perpendicular from *P* on the line *l*.

$$\frac{|P_1P|}{|PP_2|}=\frac{|P_1'P'|}{|P'P_2'|}$$

Moreover, $\frac{\overline{P_1P}}{\overline{PP_2}}$ is positive if P is between P_1 and P_2

and $\frac{\overline{P_1'P'}}{\overline{P'P_2'}}$ is positive if P′ is between P_1' and P_2'. But clearly P is between P_1' and P_2' if P is between P_1 and P_2. Thus these ratios always have the same sign and therefore,

$$\frac{\overline{P_1P}}{\overline{P'P'_2}}=\frac{\overline{P_1'P'}}{\overline{P'P'_2}}$$

Hence from our hypothesis, it follows that

$$\frac{\overline{P_1'P'}}{\overline{P'P_2'}}=r$$

and we can apply Theorem 7·5.2 (with $a=x_1$, $b=x_2$, $c=x$) to obtain $x=\frac{x_1+r\,x_2}{1+r}$.

Similarly y can be found by taking projections on the y-axis.

We assumed that the line l through P_1 and P_2 is not parallel to coordinate axes. However, the conclusion is valid in this case also

Corollory:— ·If the point P (x,y) is outside the segment $\overline{P_1\,P_2}$, then the formula will work out to be

$$x=\frac{x_1-rx_2}{1-r},\ y=\frac{y_1-ry_2}{1-r}$$

A special case of Theorem 7·5·3, which is frequently used is the case in which P is the middle point of the segment $P_1\,P_2$. This implies $r=1$, and we obtain :

Theorem 7·5·4 :— If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points on a coordinate plane, then the mid point of the segment P_1P_2 has co-ordinates

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Example 1. Given the points A (7, 5) and B(– 6, 1). Find the mid point of the segment AB.

Solution. Required mid point is

$$\left(\frac{7-6}{2}, \frac{5+1}{2}\right) \text{or} \left(\tfrac{1}{2}, 3\right).$$

Example 2. Find the coordinates of the point $P(x, y)$ which divides the line segment from $P_1(1, 7)$ to $P_2(6, -3)$ in the ratio $r = \frac{2}{3}$.

Solution. $x = \dfrac{1+(\frac{2}{3})6}{1+(\frac{2}{3})}$, $y = \dfrac{7+(\frac{2}{3})(-3)}{1+(\frac{2}{3})}$.

Thus the required point is (3,3).

Example 3. Determine the coordinates of the point $P(x, y)$ which divides the line segment from $P_1(-2, 1)$ to $P_2(3, -4)$ in the ratio $r = -8/3$.

Solution. $x = \dfrac{-2-\frac{8}{3}(3)}{1-\frac{8}{3}}$, $y = \dfrac{1-\frac{8}{3}(-4)}{1-\frac{8}{3}}$

i. e., $x = 6$, $y = -7$.

Thus the required point is(6, – 7).

Exercise 7·3

1. In the following questions, find the points of trisection of AB.

(*i*) A (4,—1), B (7,5)
(*ii*) A (—5, 2), B (8,—4)
(*iii*) A $(\frac{1}{2}, \frac{3}{2})$, B$(-\frac{5}{2}, -7)$

(*iv*) $A(\sqrt{2,-3})$, $B(-\sqrt{8,\frac{5}{2}})$.

2. Given the points A $(-1, -4)$ and B $(-6, 2)$, find the point C such that $\frac{\overline{AC}}{\overline{CB}} = \frac{1}{5}$.
3. Given the points A $(3, 7)$ and B $(-2, 4)$, find the point C such that $\frac{\overline{AC}}{\overline{CB}} = 3$.
4. If A $(-4, 3)$, B $(3, 2)$ and C are collinear, with B between A and C and, if $|AC| = 3|AB|$, find the coordinates of C.
5. If P $(-3, -6)$, Q $(4, 2)$ and R are collinear with R between P and Q, and if $|PQ| = 4|RQ|$, find the coordinates of R.
6. If P $(0, 4)$, Q $(5, 0)$, and R are collinear, with P between R and Q, and if $|RP| = 10|PQ|$, find the coordinates of R.
7. In the following questions, find the lengths of the medians of the triangle with the given vertices

   (*i*) A $(0, 0)$, B $(-2, 6)$, C $(4, 4)$

   (*ii*) A $(-3, -1)$, B $(1, 7)$, C $(9, -3)$

   (*iii*) A $(-4, 3)$, B $(3, 3)$, C $(6, -8)$

   (*iv*) A $(12, 3)$, B $(-1, -3)$, C $(-12, 8)$
8. Show that the three points, P $(-3, -4)$. Q $(2, 6)$ and R $(0, 2)$ are collinear and find the value of $\frac{\overline{PR}}{\overline{RQ}}$. Also find the value of $\frac{\overline{PQ}}{\overline{QR}}$.
9. Show that the three points A $(5, -2)$, B $(1, 2)$, and C $(-2, 5)$ are collinear and find the value of $\frac{\overline{AB}}{\overline{BC}}$ Also

find the value of $\frac{\overline{AC}}{\overline{CB}}$.

10. Let A $(3, 5)$, B $(-2, 6)$ and C $(1, -4)$ be the vertices of a triangle. If P is the mid. point of AC and Q is the mid. point of BC, verify that $|PQ| = \frac{|AB|}{2}$

11. Let O $(0, 0)$, A $(a, 0)$ and B (b, c) be the vertices of a triangle, where a, b and c are positive numbers. Find the area of the triangle whose vertices are the mid-points of the sides of OAB. Show that it is one-fourth the area of the given triangle OAB.

12. $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the vertices of a triangle, prove that the point

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

is the point of trisection of the three medians of the triangle.

13. Find the coordinates of the centre of the circle touching internally the sides of the triangle whose vertices are $(4, -2)$, $(-2, 4)$ and $(5, 5)$.

ANSWERS

1. (*i*) $(5, 1)$, $(6, 3)$

   (*ii*) $\left(-\frac{2}{3}, 0\right)$, $\left(\frac{11}{3}, -2\right)$

   (*iii*) $\left(-\frac{1}{2}, -\frac{4}{3}\right)$, $\left(-\frac{3}{2}, -\frac{25}{6}\right)$

   (*iv*) $\left(0, -\frac{7}{6}\right)$, $\left(-\sqrt{2}, \frac{2}{3}\right)$

2. $\left(\frac{-11}{6}, -3\right)$

3. $\left(-\frac{3}{4}, \frac{19}{4}\right)$

4. $(17, 0)$

5. $\left(\frac{9}{4}, 0\right)$

6. $(-50, 44)$

7. (*i*) $\sqrt{26}$, $4\sqrt{2}$, $\sqrt{26}$

   (*ii*) $\sqrt{73}$, $\sqrt{85}$, $\sqrt{136}$

   (*iii*) $\frac{\sqrt{410}}{2}$, $\frac{\sqrt{137}}{2}$, $\frac{\sqrt{653}}{2}$

   (*iv*) $\frac{\sqrt{1370}}{2}$, $\frac{\sqrt{293}}{2}$, $\frac{\sqrt{1481}}{2}$

8. $\frac{3}{2}; -\frac{5}{2}$

9. $\frac{4}{3}; -\frac{7}{3}$ 13. $\frac{5}{2}, \frac{5}{2}$

7.6. *Angle from one line to another.* We know that two straight lines in a plane either intersect or are parallel. The angle from one line to another is defined as follows :—

Definition. I— If l_1 and l_2 are intersecting lines, the angle from l_1 to l_2 is the angle of least positive measure having its initial arm on l_1 and terminal arm on l_2. In case l_1 and l_2 are parallel, the angle from l_1 to l_2 is defined as the angle of zero measure.

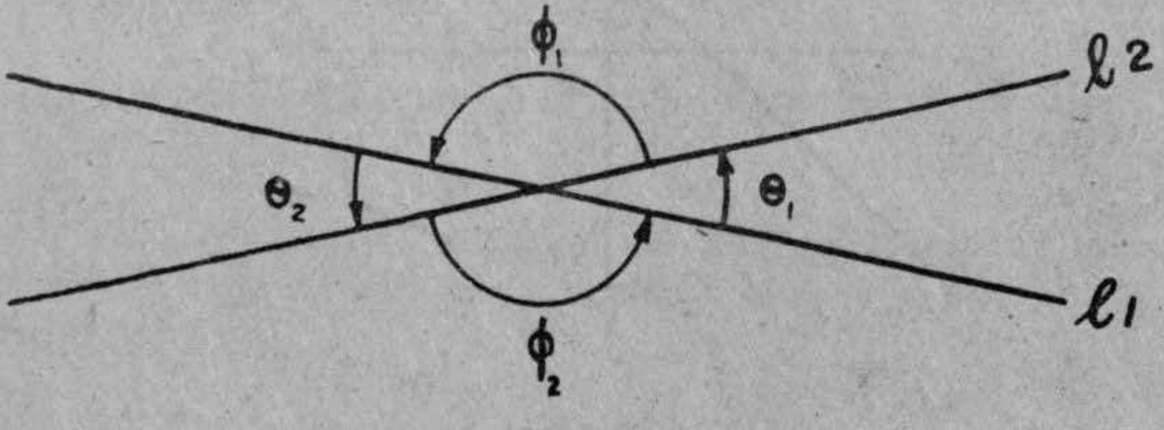


Fig · 7.11

In fact there are two angles from l_1 to l_2 that satisfy the definition, namely the angles θ_1 and θ_2 in Fig 7·11, similarly there are two angles from l_3 to l_1 that is, $\varnothing_1$ and $\varnothing_2$ in the figure. Since $\theta_1 = \theta_2$ and $\varnothing_1 = \varnothing_2$ it obviously makes no difference whether we consider θ_1 or θ_2 as the angle from l_1 to l_2, and whether we consider $\varnothing_1$ or $\varnothing_2$ as the angle from l_2 to l_1.

From the definition it follows that the measure of the angle from any line to any other line is either 0°, or positive ($<180^\circ$).

Notice also that if l_1 and l_2 are not parallel, the sum of the measures of the angles from l_1 to l_2 and from l_2 to l_1 is 180° (in Fig 7·11 $\theta_1 + \varnothing_1 = 180^\circ$).

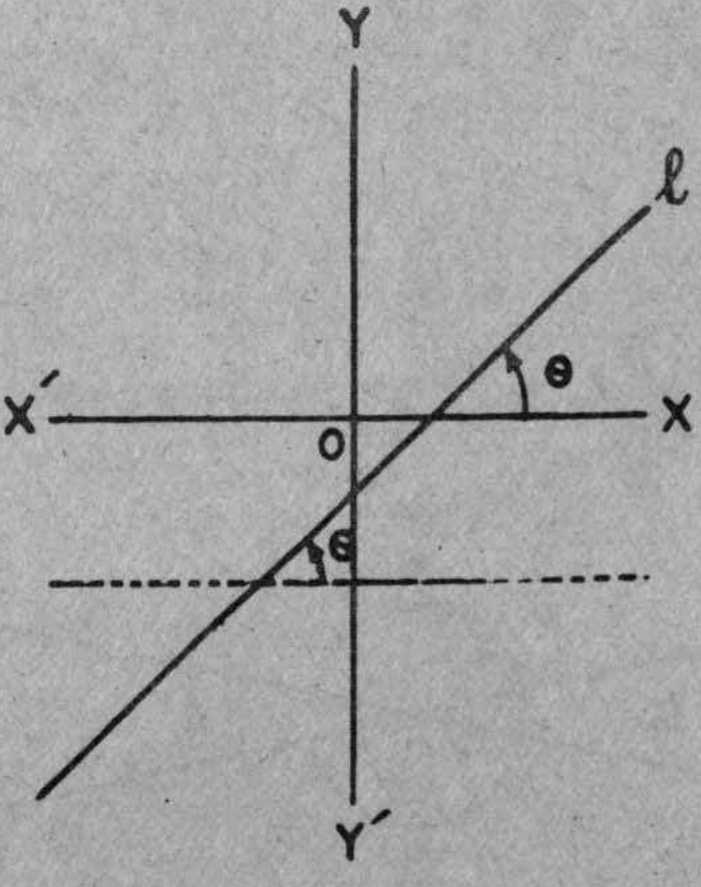


Fig. 7.12

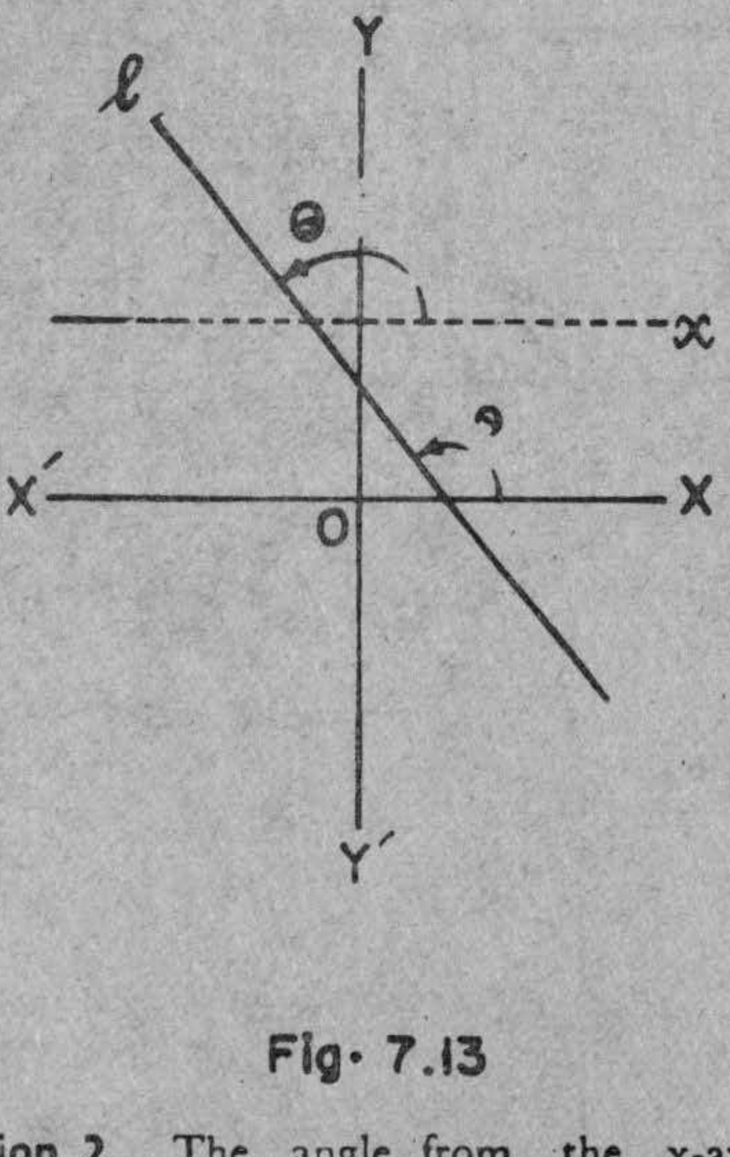


Fig. 7.13

Definition 2. The angle from the x-axis to a line l is called the inclination of the line l.

In figures 7·12 and 7·13 angle θ is the inclination of the line.

It is important to observe that if x' is a line parallel to x-axis, the angle from x' to l is also the inclination of l.

7·7. *The inclination of the line passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$.*

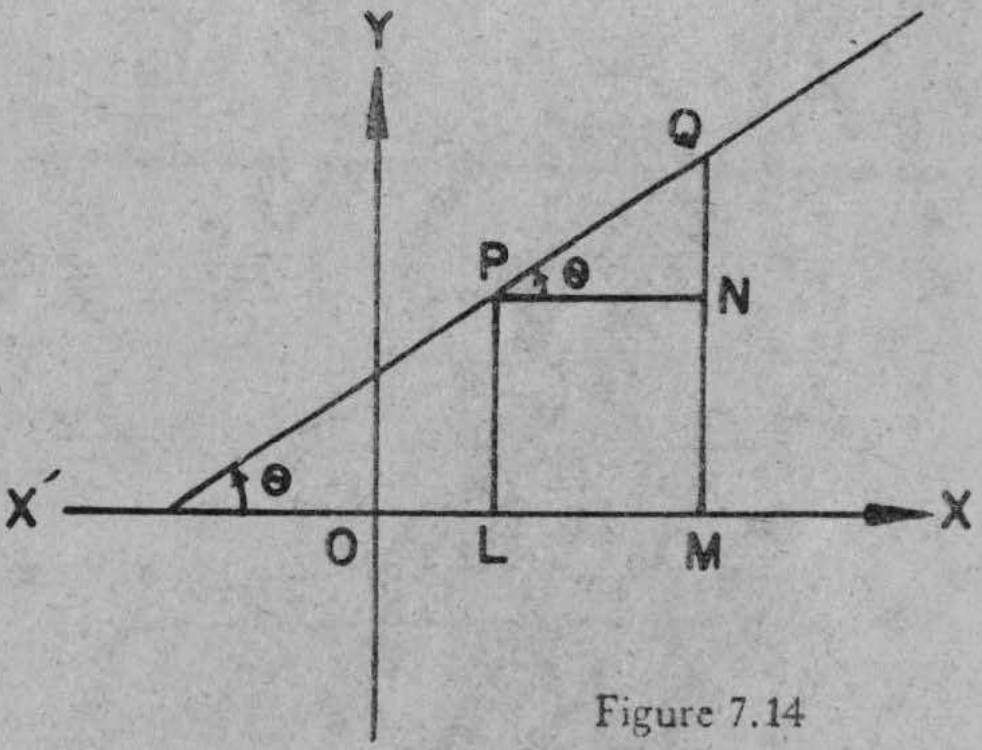

Y
Q
P
θ
N
θ
X′
O
L
M
X
Y

Figure 7.14

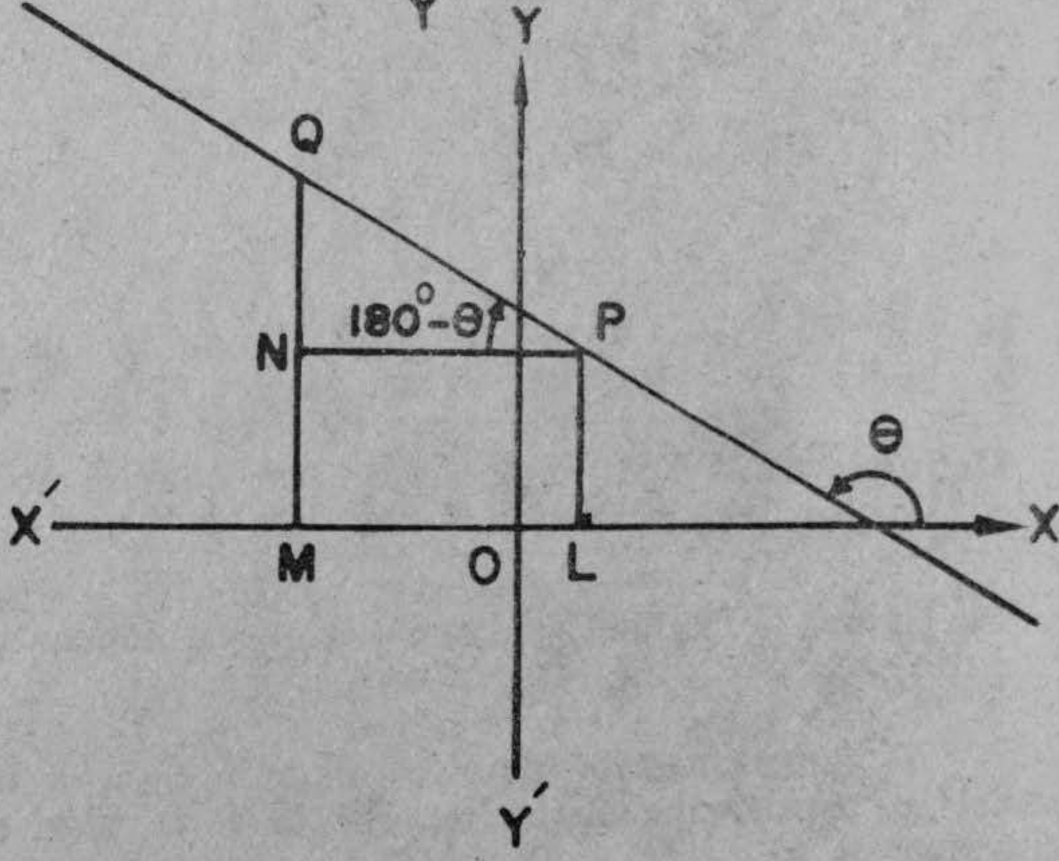

Y
Q
180°−θ
P
N
θ
X′
M
O
L
X
Y′

Figure 7.15

From Figure 7.14 we have

$$\text{Tan } \theta = \frac{\overline{NQ}}{\overline{PN}} = \frac{y_2 - y_1}{x_2 - x_1}$$

From Figure 7.15 we have

$$\text{Tan } (180 - \theta) = \frac{\overline{NQ}}{\overline{NP}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\therefore \text{ Tan } \theta = \frac{y_2 - y_1}{x_2 - x_1} .$$

Thus in all cases the inclination of line *l* is given by

$$\theta = \text{Tan}^{-1} \frac{y_2 - y_1}{x_2 - x_1} = \text{Tan}^{-1} \frac{\text{difference of ordinates}}{\text{difference of abscissae}}$$

The above result holds for any two points on the line *l*.

7.8. *Slope of a line* :

Definition. Tan θ, where θ is the inclination of a line is called the slope or gradient of the line and is denoted by *m*.

Note 1. The slope of x-axis is zero and the slope of y—axis is not defined.

Note 2. As x-axis is generally horizontal, any line parallel to x—axis is horizontal and any line parallel to y—axis is vertical.

7.9. ***Parallel and Perpendicular lines*** :—

7.9.1. ***Theorem*****—Two straight lines are parallel if their slopes are equal.**

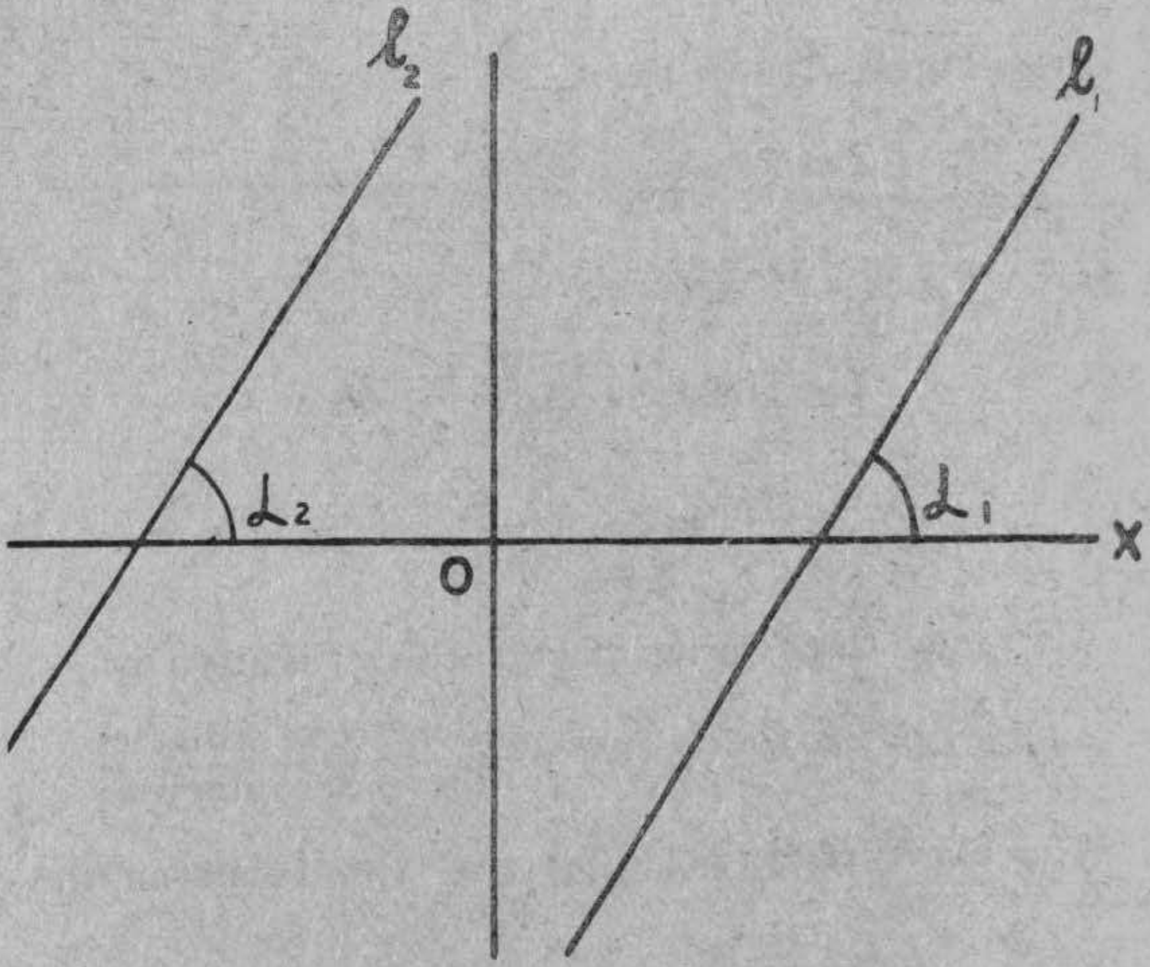


Fig. 7.16

Proof : Let l_1 be parallel to l_2. Let their inclinations be α_1 and α_2 respectively.

Since $l_1 // l_2$,

$\alpha_1 = \alpha_2$ (corresponding angles) and

hence Tan α_1 = Tan α_2 *i.e* slope of l_2 = slope of l_1.

7.9.2 *Theorem*. Two straight lines are perpendicular or orthogonal if the product of their slopes is -1.

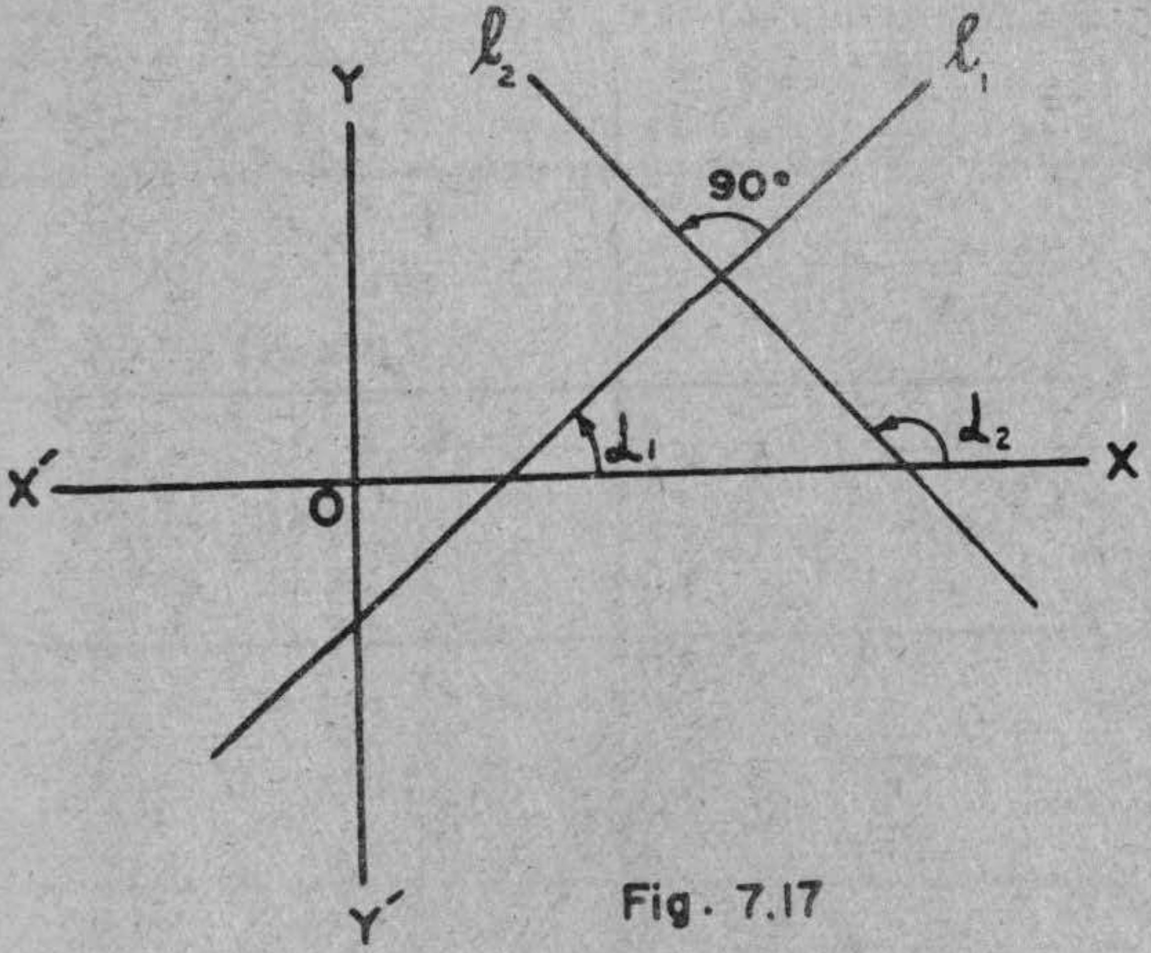


Fig. 7.17

Proof. Let l_1 and l_2 be two perpendicular lines. Let their inclinations be α_1 and α_2 respectively.

Since $l_1 \perp l_2$,

$\alpha_2 = 90^\circ + \alpha_1$ and

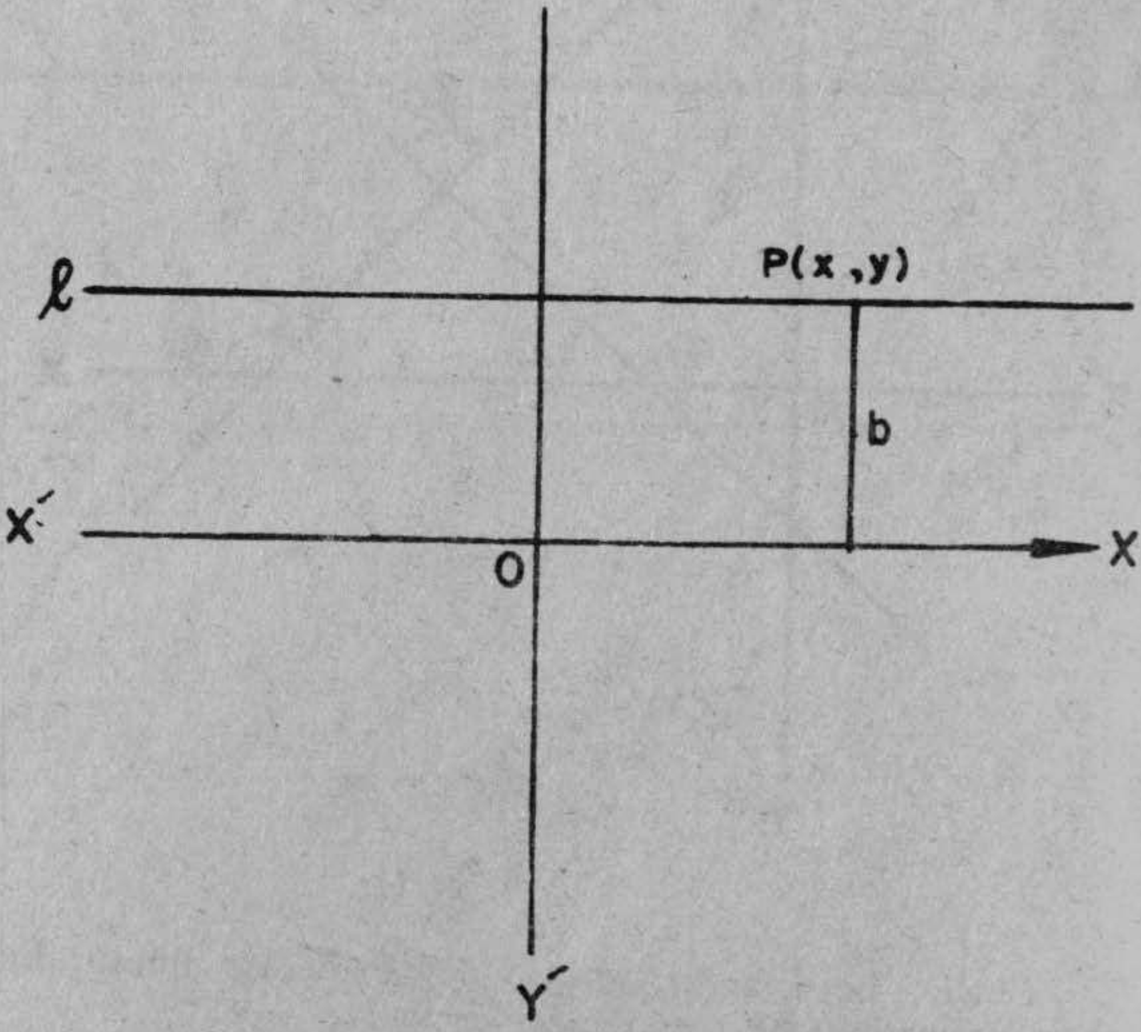
hence $\text{Tan}\ \alpha_2 = \text{Tan}\ (90 + \alpha_1) = -\text{Cot}\ \alpha_1 = -\dfrac{1}{\text{Tan}\ \alpha_1}$.

Thus $\text{Tan}\ \alpha_1\ \text{Tan}\ \alpha_2 = -1$

i.e. product of the slopes of l_1 and l_2 is -1.

7.10. *Equation of a straight line—*

I. Lines Parallel to Axes.



F·g 7.18

Let *l* be a straight line prarallel to x-axis at a distance *b* from it.

A point P $(x,)$ lies on it if $y = b$. (1)

This is, therefore, the equation of a line parallel to **x-axis** at a distance of b units from it.

Similarly the equation of a line parallel to **y-axis** at a distance of a units from it is $x = a$. ....(2)

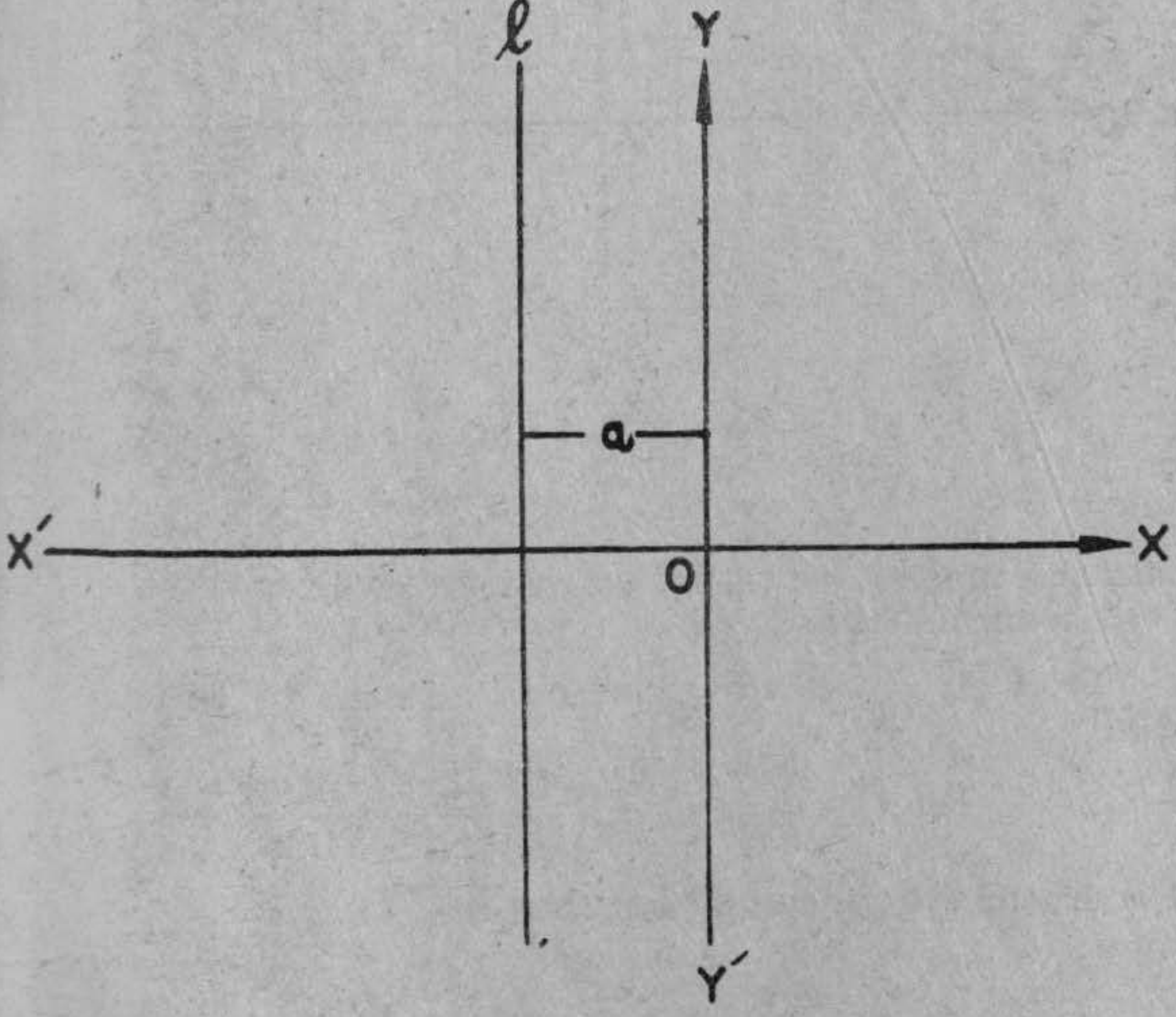


Fig. 7.19

Note : The equation of **x-axis** is $y = 0$. The equation of **y-axis** is $x = 0$

II. *Slope—Intercept Form.*

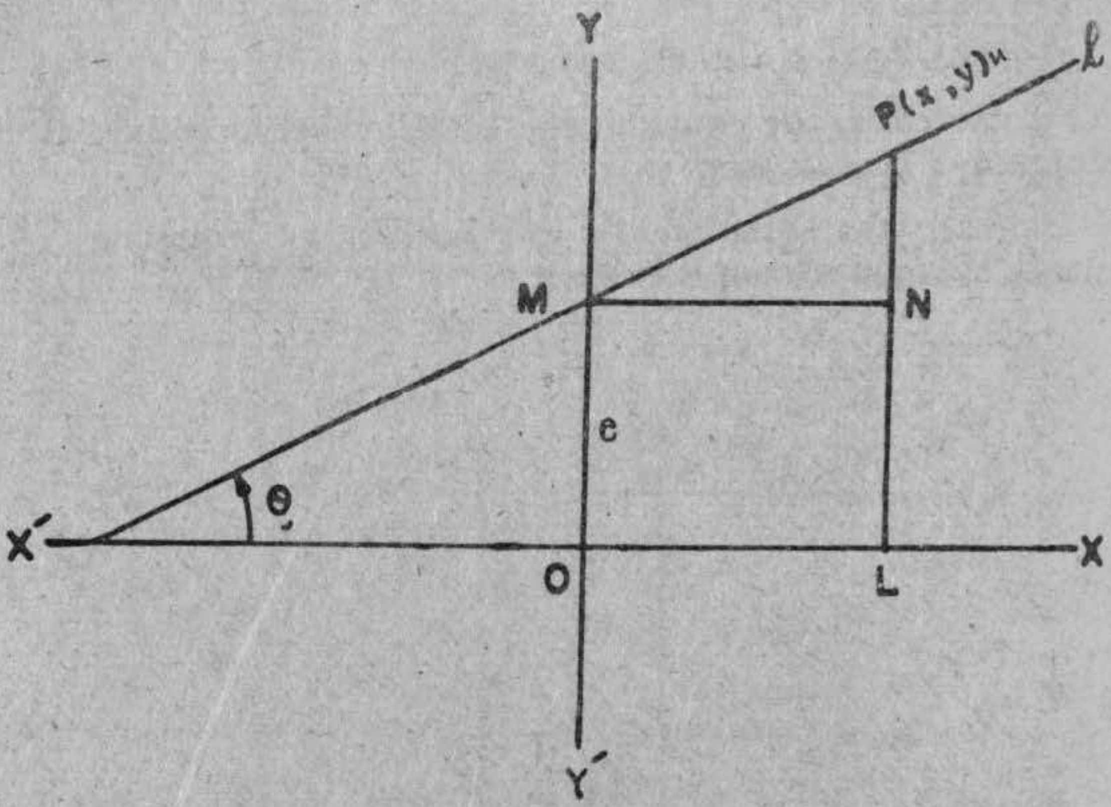


Fig. 7.20

Let l be a straight line cutting y-axis at M such that OM $= c$. Let θ be the inclination of l.

Let P (x, y) be any point on From the triangle MNP

$$\text{Tan}\ \theta = \frac{\overline{NP}}{\overline{MN}} = \frac{\overline{LP}-\overline{LN}}{\overline{OL}} = \frac{\overline{LP}-\overline{OM}}{\overline{OL}} = \frac{y-c}{x}.$$

If m denotes the slope of the line then $m = \text{Tan}\ \theta$

$$\therefore \quad m = \frac{y-c}{x}$$

or $mx = y - c$

i.e. $y = mx + c$ (3)

is the equation of the line in terms of its slope m and intercept c on y-axis.

Deductions.

1. *Equation of a line passing through origin.*

If the line l passes through the origin O, $c=0$ and equation (3) reduces to $y=mx$.

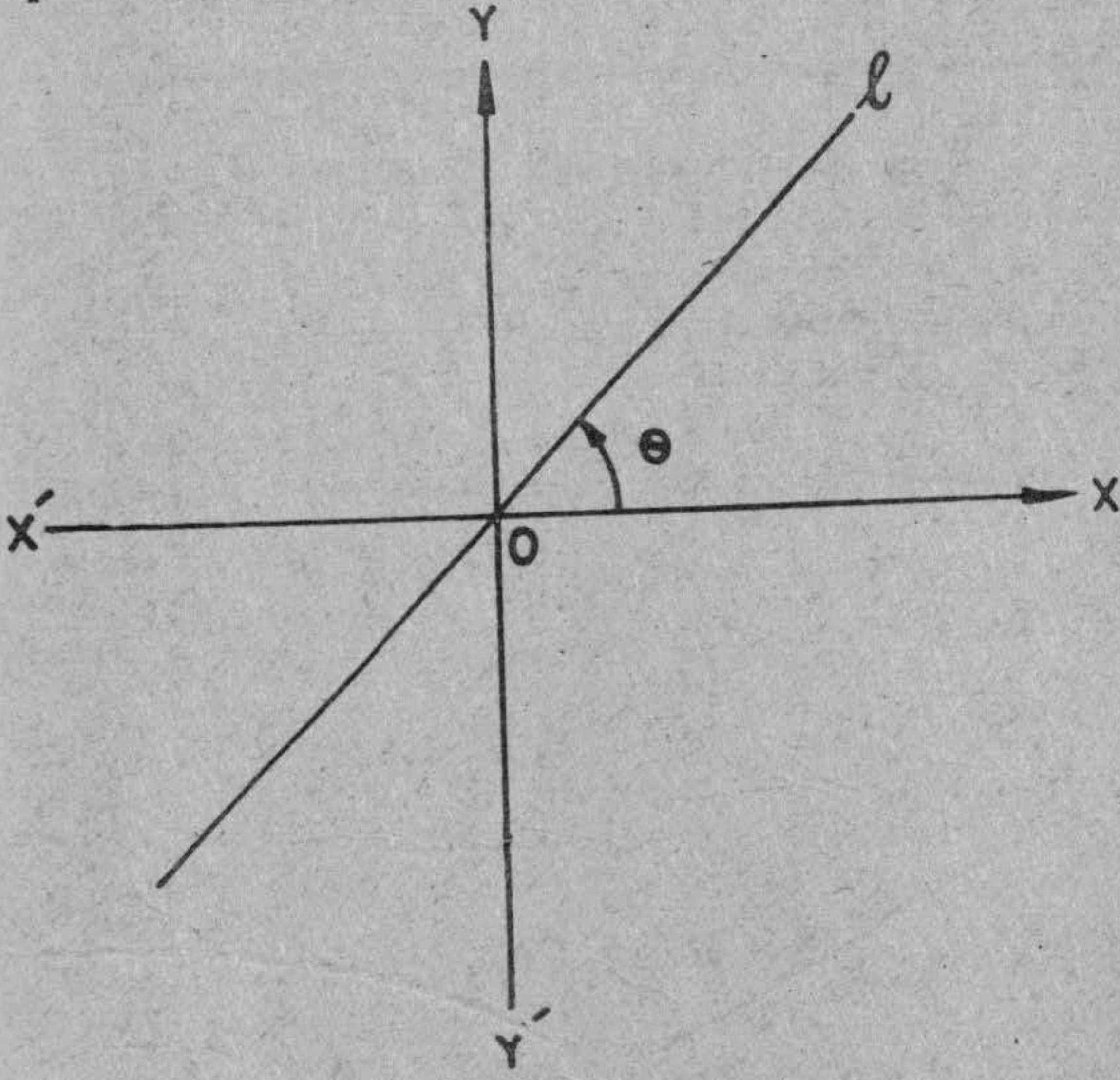


Fig. 7.21

2. *Equation of a line parallel to x-axis.*

If the line l is parallel to x-axis, $m=0$ and equation (3) reduces to

$$y=c$$

where c is the distance of the line from x-axis.

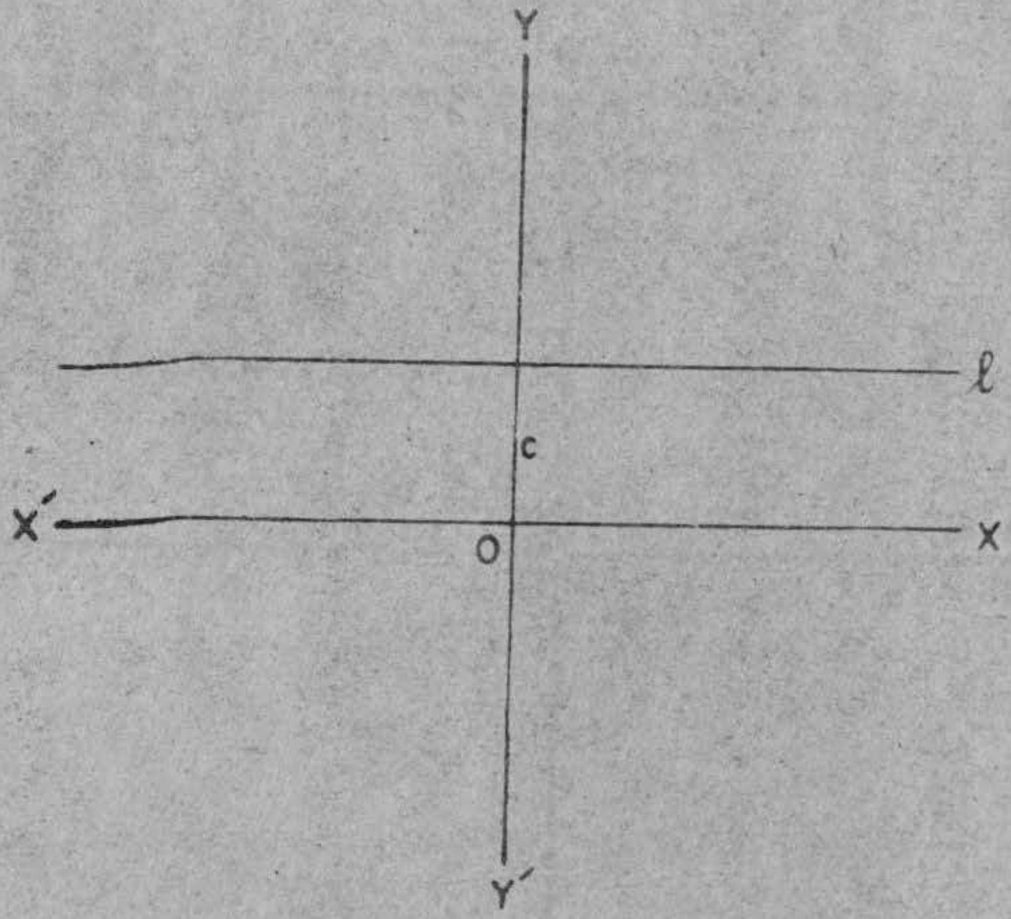


Fig. 7.22

3. *The equation of a straight line passing through* (x_1, y_1) *with slope m.*

If (x_1, y_1) lies on (3) then

$$y_1 = mx_1 + c. \qquad \text{(A)}$$

Subtracting (A) from (3) we have

$$y - y_1 = m(x - x_1) \qquad (4)$$

as the required equation.

4. *The equation of a straight line passing through* (x_1, y_1) *and* (x_2, y_2).

The slope of the line is

$$\frac{y_2-y_1}{x_2-x_1}=m.$$

$\therefore$ From (4) we have the equation of the line passing through two points as

$$y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad (5)$$

$$\text{or } \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

III. *Intercept Form.* Let l be a straight line having x-intercept equal to a and y-intercept equal to b. *i.e.* OA $=a$, OB $=b$.

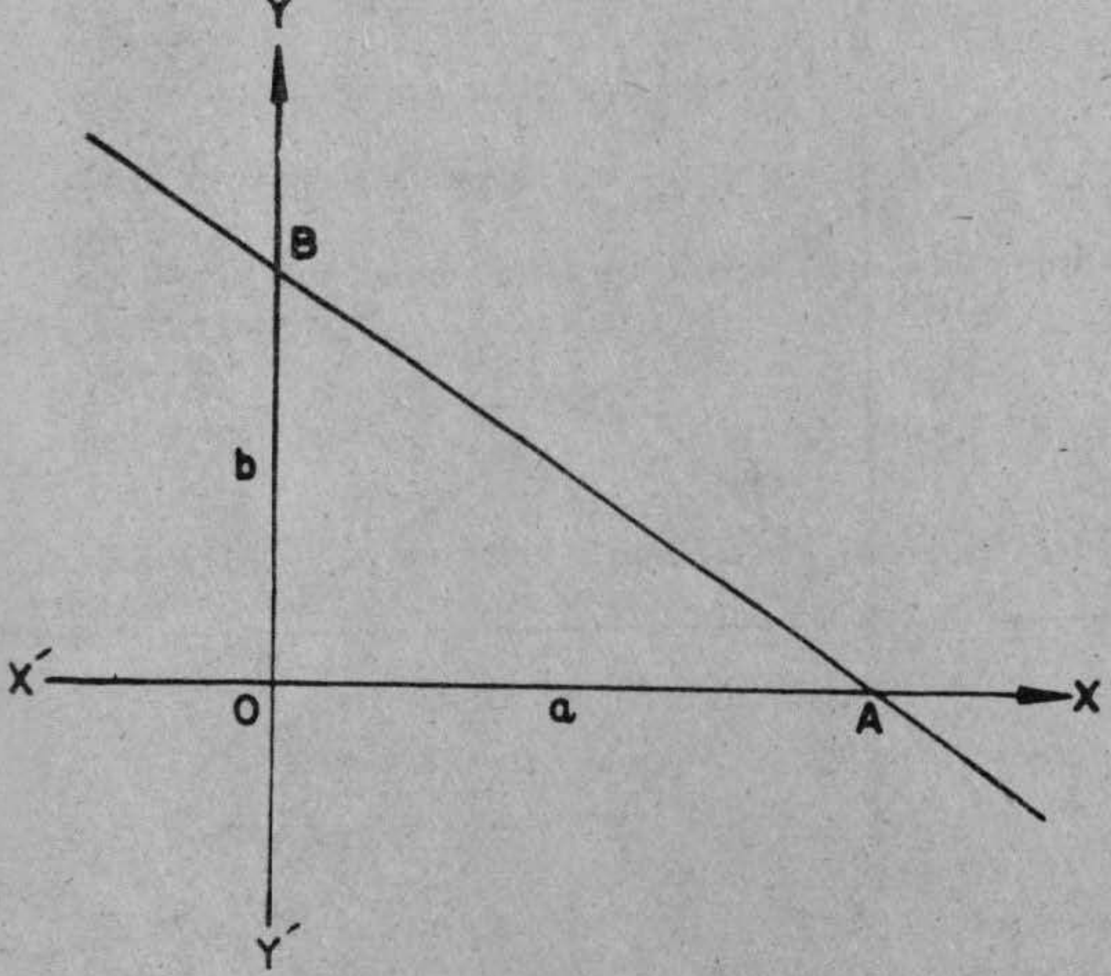


Fig. 7.23

Therefore the line passes through **A** $(a, 0)$ and **B** $(0, b)$ From (5) the equation of the line is

$$y-0 = \frac{b-0}{0-a}(x-a)$$

$$\text{or } y = \frac{-b}{a}(x-a)$$

$$\text{or } \frac{y}{b} = \frac{-x}{a}+1$$

$$\text{or } \frac{x}{a}+\frac{y}{b}=1. \qquad (6)$$

IV. *Perpendicular Form.* Let l be a straight line, and p the length of the perpendicular from the origin **O** on the line and α the inclination of the perpendicular.

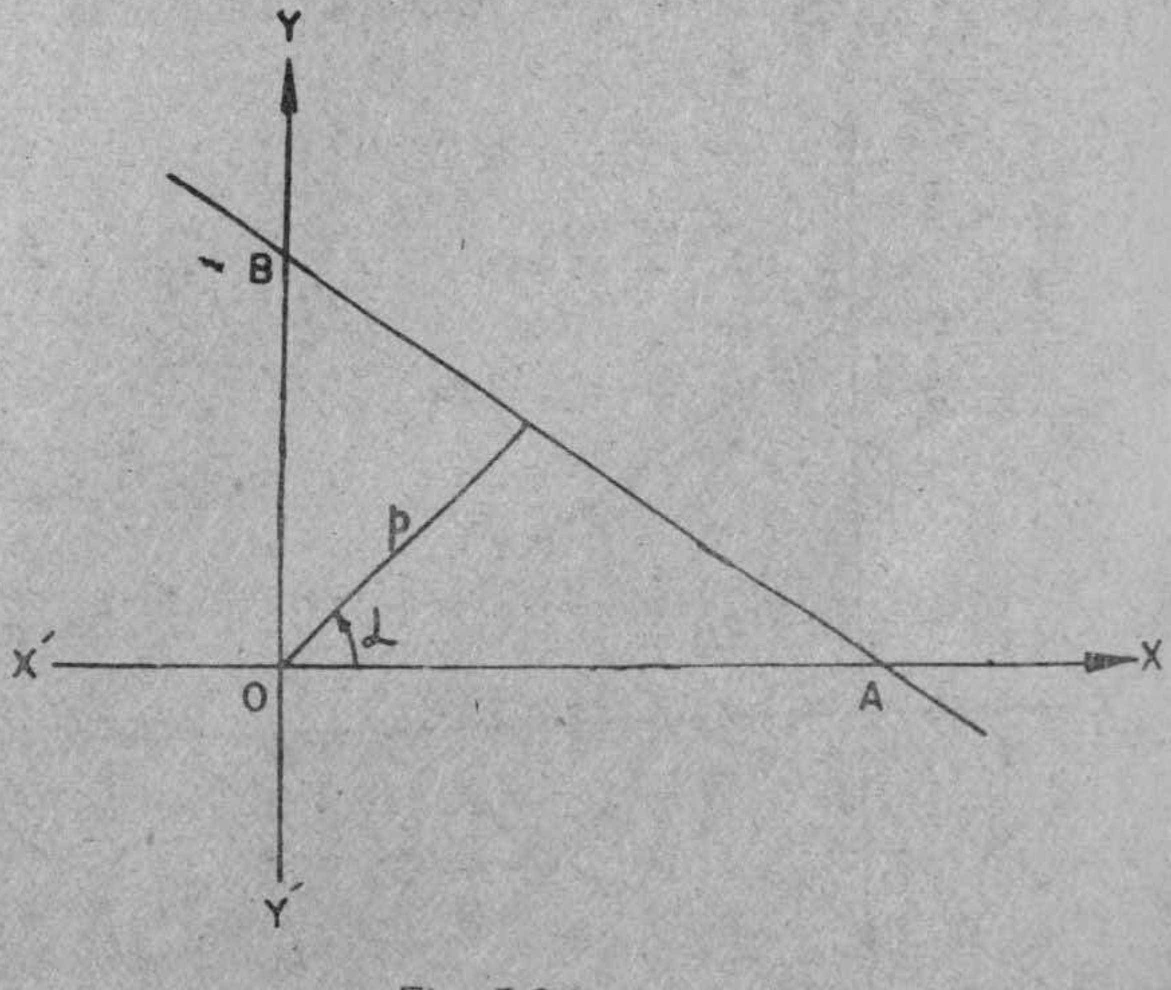


Fig. 7.24

Now $\text{Cos}\,\alpha = \dfrac{p}{OA}$ and $\text{Sin}\,\alpha = \dfrac{p}{OB}$

$\therefore \quad OA = \dfrac{p}{\text{Cos}\,\alpha}, \quad OB = \dfrac{p}{\text{Sin}\,\alpha}.$

From (6) the equation of the line is

$$\frac{x}{\dfrac{p}{\text{Cos}\,\alpha}} + \frac{y}{\dfrac{p}{\text{Sin}\,\alpha}} = 1$$

or $x\,\text{Cos}\,\alpha + y\,\text{Sin}\,\alpha = p$.

Example 1. Find the equation of the line which passes through the point $(-5, -3)$ and is parallel to the x-axis.

Solution. Any line parallel to x-axis is of the form $y = b$.

Since $(-5, -3)$ lies on it, $-3 = b$.

Thus the required line is $y = -3$ or $y + 3 = 0$.

Example 2. Find the slope of the line which passes through the points $P_1\,(3, -2)$ and $P_2\,(-2, 5)$.

Solution. $m = \dfrac{5-(-2)}{-2-3} = \dfrac{7}{-5} = -\dfrac{7}{5}$.

Example 3. Find the equation of the line whose y-intercept is -7 and whose slope is $\frac{1}{2}$.

Solution. Using (3), the required equation is

$$y = \tfrac{1}{2}x - 7$$

or $\quad x - 2y - 14 = 0$.

Example 4. Through the point $P\,(-1, 2)$ draw the line whose slope is $-\frac{3}{2}$. Also find its equation.

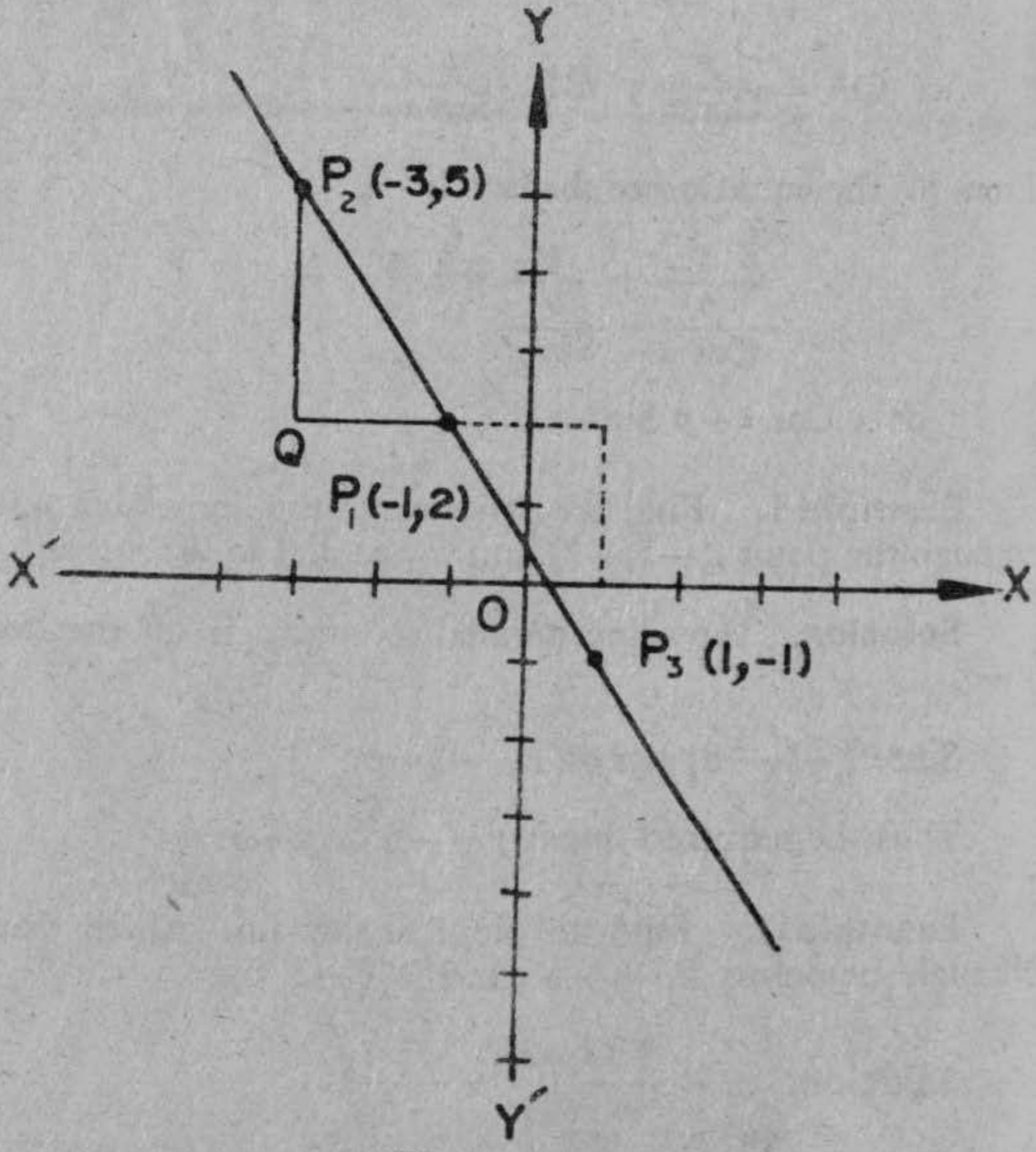

Y
P_2 (-3,5)
Q
P_1(-1,2)
X′
X
O
P_3 (1,-1)
Y′

Fig. 7.25

Solution. The equation of the required line is

$$y-2=-\tfrac{3}{2}(x+1)$$

or $2y-4=-3x-3$, or $3x+2y-1=0$.

To draw the graph we proceed as follows.

Since $-\frac{3}{2}$ may be considered as $\frac{3}{-2}$ then, as in figure 7.25,

take $P_1Q=-2$ and $QP_2=3$. Then by counting 2 units to the left from P_1 and 3 units up, the coordinates of P_3 are found to be $(-3, 5)$ and the line drawn through P_1 and P_2 is the required line.

This construction may also be made as follows :

Since $-\frac{3}{2}$ may be considered as $\frac{-3}{2}$ then count 2 units to the right of P_1 and 3 units down to locate the point P_3 whose coordinates are found to be $(1,-1)$, as in figure 7.25. Then join the two points, P_1 and P_3 to get the required line.

Example 5. Find the line which passes through the point $(6,-4)$ and whose slope is $-\frac{1}{3}$.

Solution. Required equation is $y-(-4)=-\frac{1}{3}(x-6)$ [From (4)].

$$i.e. \quad x+3y+6=0.$$

Example 6. Find the equation of the line which passes through the points $(1,-5)$ and $(-4, 4)$.

Solution. Since the slope of the line is

$$m=\frac{4+5}{-4-1}=-\frac{9}{5},$$

the required equation is

$$y-(-5)=-\frac{9}{5}(x-1)$$

$$i.e \quad 5y+25=-9x+9$$

$$\text{or} \quad 9x+5y+16=0.$$

Exercise 7.4

1. Find the equation of the straight line which satisfies each of the following conditions :—

(*i*) parallel to x-axis and 7 units below it.

(*ii*) parallel to y-axis and 3 units to the right of it.

(*iii*) through (7, —3) and parallel to x-axis.

(*iv*) through (—7, 5) and parallel to y-axis.

2. Find the slopes of the lines which pass through the following pairs of points :—

(*i*) (3, —1) and (5, —4)

(*ii*) (—4, 2) and (3, 5)

(*iii*) (0, 0) and (—4, 3).

3. Draw the following straight lines and find their equations :

(*i*) through (3, 2) with slope 2

(*ii*) through (—4, 3) with slope —1

(*iii*) through (4, —3) with slope $-\frac{2}{3}$

(*iv*) through (2, —3) with slope $\frac{1}{2}$

4. Find the equation of the straight line determined by each of the following sets of conditions :—

(*i*) y-intercept 5 ; slope 3

(*ii*) y-intercept 4 ; slope —1

(*iii*) y-intercept 3 ; slope $-\frac{2}{3}$.

5. Show that the points (2, 6), (—8, 1) and (—2, 4) lie on the same straight line.

6. Find the equations of the following lines :—

(*i*) through (2, 5) with slope 2

(*ii*) through (4, —1) with slope 5

(*iii*) through (0, —3) with slope $-\frac{2}{3}$

(*iv*) through the origin with slope $-\frac{7}{9}$.

7. Find the equations of the following lines :—

(*i*) through (5, 7) and (1, 1)

(*ii*) through (—2,—4) and (1, 3)

(*iii*) through (5, 0) and (—2, 3)

(*iv*) through $(a, 0)$ and $(0, b)$.

ANSWERS

1. (*i*) $y+7=0$ (*ii*) $x-3=0$
(*iii*) $y+3=0$ (*iv*) $x+7=0$

2. (*i*) $-\frac{3}{2}$ (*ii*) $\frac{3}{7}$
(*iii*) $-\frac{3}{4}$

3. (*i*) $y-2=2(x-3)$ (*ii*) $y-3=-1(x+4)$
(*iii*) $y+3=-\frac{2}{3}(x-4)$ (*iv*) $y+3=\frac{1}{2}(x-2)$

4. (*i*) $3x-y+5=0$ (*ii*) $x+y-4=0$
(*iii*) $2x+3y-9=0$

6. (*i*) $2x-y+1=0$ (*ii*) $5x-y-21=0$
(*iii*) $2x+3y+9=0$ (*iv*) $7x+9y=0$

7. (*i*) $3x-2y-1=0$ (*ii*) $7x-3y+2=0$
(*iii*) $3x+7y-15=0$ (*iv*) $bx+ay-ab=0$

7.11 *Linear Equation*. An equation of the first degree in the variables x and y is called a linear equation in these variables.

The general form of a linear equation is

$$Ax+By+C=0$$

where A. B, C are constants and A and B are not both zeros. We observe that the different forms of the equation of a straight line derived in § 7.10 are all linear.

Conversely, we have

7.11.1. *Theorem.* Every linear equation represents a straight line.

Proof : Let $Ax+By+C=0$ ....(1)
be the given linear equation.

If $B=0$ then $A\neq 0$ and (1) reduces to

$$Ax+C=0$$

$$\text{or}\quad x=\frac{-C}{A}$$

which is a straight line parallel to y-axis.

Similarly if $A=0$, $B\neq 0$ then (1) is a straight line parallel to x-axis.

If $A\neq 0$ and $B\neq 0$ then (1) can be written as

$$y=-\frac{A}{B}x-\frac{C}{B}$$

which is of the slope-intercept form and represents a straight line with slope $m=\frac{-A}{B}$ and y-intercept $c=\frac{-C}{B}$.

Note : The equation (1) can be reduced to the intercept form and the perpendicular form under suitable conditions.

Example 1. Reduce the equation $5x+3y-15=0$ to

(*i*) slope-intercept form

(*ii*) intercept form

(*iii*) perpendicular form.

Solution. (*i*) The equation can be written as

$$3y=-5x+15$$

$$\text{or } y=-\tfrac{5}{3}x+5$$

which is slope-intercept form.

$$\text{Slope}=-\tfrac{5}{3}.\ \text{y-intercept}=5.$$

(*ii*) The equation can be written as

$$5x+3y=15$$

$$\text{or}\quad \frac{x}{3}+\frac{y}{5}=1$$

which is the intercept form,

where x-intercept $=3$, y-intercept $=5$.

(*iii*) The equation can be written as

$$5x+3y=15.$$

Comparing this equation with

$$x \operatorname{Cos} \alpha + y \operatorname{Sin} \alpha = p$$

We have

$$\frac{\operatorname{Cos} \alpha}{5}=\frac{\operatorname{Sin} \alpha}{3}=\frac{p}{15}=k$$

$$\therefore \operatorname{Cos} \alpha = 5k, \operatorname{Sin} \alpha = 3k.$$

Squaring and adding

$$1=25k^2+9k^2$$

or $k^2=\frac{1}{34}, k=\pm\frac{1}{\sqrt{34}}$.

We select that sign in the value of k which renders p positive. In the present Case

$$k=\frac{1}{\sqrt{34}}$$

$$\therefore \operatorname{Cos} \alpha=\frac{5}{\sqrt{34}},$$

$$\operatorname{Sin} \alpha=\frac{3}{\sqrt{34}},$$

$$p=\frac{15}{\sqrt{34}},$$

$\therefore$ the equation in perpendicular form is

$$\frac{5}{\sqrt{34}}x+\frac{3}{\sqrt{34}}y=\frac{15}{\sqrt{34}}.$$

Example 2. Are the lines $2x-3y-2=0$ and $4x-6y+5=0$ parallel ?

Solution. If $2x-3y-2=0$, $y=\frac{2}{3}x-\frac{2}{3}$.
Therefore its slope $m_1=\frac{2}{3}$.

Also, if $4x-6y+5=0$, $y=\frac{2}{3}x+\frac{5}{6}$.

Therefore its slope $m_2=\frac{2}{3}$.

Since the slopes are equal, the lines are parallel.

Example 3. Find the equation of the line through $(3,-1)$ which is parallel to the line $2x+3y-5=0$.

Solution. By solving $2x+3y-5=0$ for y, we see that the slope of the given line is $-\frac{2}{3}$. Hence equation of the desired line is

$$y+1=-\frac{2}{3}(x-3)$$

$$\text{or}\quad 2x+3y-3=0.$$

Example 4. Find the equation of the line through $(3,-1)$ and perpendicular to the line $2x+3y+5=0$.

Solution. The equation $2x+3y+5=0$ in the slope form is

$$y=-\frac{2}{3}x-\frac{5}{3}$$

$$\therefore\quad \text{slope of the line}=-\frac{2}{3}.$$

Slope of the line perpendicular to it $=\frac{3}{2}$.

Line passing through $(3,-1)$ and having slope $\frac{3}{2}$ is

$$y+1=\frac{3}{2}(x-3)$$

$$\text{or}\quad 3x-2y-11=0.$$

Exercise 7.5

1. Draw the following lines using their slope and y-intercept.

(*i*) $3x+y-2=0$
(*ii*) $3x+4y-10=0$
(*iii*) $4x-3y+14=0$
(*iv*) $2x+3y+4=0$

2. Reduce the equations in problem 1 to intercept form and perpendicular form.

3. Determine which of the following pairs of lines are (*a*) parallel, (*b*) perpendicular (*c*) neither.

(*i*) $x-y+3=0$; $x+y-7=0$

(*ii*) $4x-y+5=0$; $8x-2y-9=0$

(*iii*) $5x+3y-8=0$; $3x-5y+3=0$

(*iv*) $x+3y-9=0$; $2x-y+4=0$

(*v*) $3x-4y-12=0$; $8x+6y-1=0$

(*vi*) $5x+7y+3=0$; $6x-3y+2=0$.

4. Find the equations of each of the following straight lines :—

(*i*) Through $(3, 1)$ and parallel to $2x+5y-4=0$

(*ii*) Through $(-1,-4)$ and perpendicular to $2x-3y-4=0$

(*iii*) Through $(7, 1)$ and perpendicular to $7x+3y+6=0$

(*iv*) Through $(5,-2)$ and parallel to the line through the points $(1, 4)$ and $(-2,-7)$.

(*v*) Through $(-1, 7)$ and perpendicular to the line through the points $(2, 3)$ and $(0, -4)$.

5. Prove by use of slopes that the points $(4, 2)$, $(-3, 7)$ and $(-1, -5)$ are the vertices of a right angled triangle.

6. Prove by use of slopes that the points $(-4, 3)$, $(-5,-3)$, $(2,-1)$ and $(3, 5)$ are the vertices of a parallelogram.

7. Find the equation of the perpendicular bisector of the line segment joining the points $(4, 7)$ and $(-2, 1)$.

ANSWERS

1. (*i*) Slope -3, intercept 2

(*ii*) Slope $-\frac{3}{4}$, intercept $\frac{5}{2}$

(*iii*) Slope $\frac{4}{3}$, intercept $\frac{14}{3}$

(*iv*) Slope $-\frac{2}{3}$, intercept $-\frac{4}{3}$.

2. (*i*) Intercept form $\frac{x}{\frac{2}{3}}+\frac{y}{2}=1$;

Perpendicular form $\frac{3}{\sqrt{10}}x+\frac{1}{\sqrt{10}}y=\frac{2}{\sqrt{10}}$

(*ii*) Intercept form $\frac{x}{\frac{10}{3}}+\frac{y}{\frac{5}{2}}=1$

Perpendicular form $\frac{3}{5}x+\frac{4}{5}y=2$

(*iii*) Intercept form $\frac{x}{-\frac{14}{4}}+\frac{y}{\frac{14}{3}}=1$

Perpendicular form $-\frac{4}{5}x+\frac{3}{5}y=\frac{14}{5}$

(*iv*) Intercept form $\frac{x}{-2}+\frac{y}{-\frac{4}{3}}=1$

Perpendicular form $-\frac{2}{\sqrt{13}}x-\frac{3}{\sqrt{13}}y=\frac{4}{\sqrt{13}}$

3. (*i*) Perpendicular
   (*ii*) Parallel
   (*iii*) Perpendicular
   (*iv*) neither
   (*v*) Perpendicular
   (*vi*) neither

4. (*i*) $2x+5y-11=0$
   (*ii*) $3x+2y+11=0$
   (*iii*) $3x-7y-14=0$
   (*iv*) $11x-3y-61=0$
   (*v*) $2x+7y-47-0$

7. $x+y-5=0$.

7.12. *The intersection of two straight lines.*

Let $l_1 \equiv a_1 x + b_1 y + c_1 = 0$ (1)

$l_2 \equiv a_2 x + b_2 y + c_2 = 0$ (2)

be the two given lines. Their point of intersection can be found by solving (1) and (2) simultaneously as :

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1}$$

or $x = \dfrac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \; y = \dfrac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}$

Hence point of intersection of l_1 and l_2 is

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right)$$

If $a_1 b_2 - a_2 b_1 = 0$ then the above coordinates have no meaning and the lines do not intersect but are parallel.

Hence l_1 and l_2 intersect if

$$a_1 b_2 - a_2 b_1 \neq 0.$$

Definition. Lines intersecting at the same point are called *concurrent* lines and the point of intersection is known as the *point of concurrency*.

7.12.1. *Condition of Concurrency of Three Lines.*

The three lines $l_1 = a_1 x + b_1 y + c_1 = 0,$

$l_2 = a_2 x + b_2 y + c_2 = 0,$

$l_3 = a_3 x + b_3 y + c_3 = 0,$

will be concurrent if the coordinates of the point of interection of l_1 and l_2 satisfy the equation of l_3.

By 7.12, the coordinates of the point of intersection of l_1 and l_2 are

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right)$$

Putting them in l_3 we get

$$a_3\left(\frac{b_1 c_2-b_2 c_1}{a_1 b_2-a_2 b_1}\right)+b_3\left(\frac{c_1 a_2-c_2 a_1}{a_1 b_2-a_2 b_1}\right)+c_3=0;$$

on simplification this reduces to

$$a_3 (b_1 c_2-b_2 c_1)+b_3 (c_1 a_2-c_2 a_1)+c_3 (a_1 b_2-a_2 b_1)=0,$$

which is the condition of concurrency of the three lines l_1, l_2, l_3.

This condition can be written in determinant form as follows :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Example. Prove that the lines

$$2x-3y-7=0,$$

$$3x-4y-13=0,$$

$$8x-11y-33=0,$$

are concurrent. Also find the point of concurrency.

Solution. The determinant of their coefficients

$$= \begin{vmatrix} 2 & -3 & -7 \\ 3 & -4 & -13 \\ 8 & -11 & -33 \end{vmatrix}$$

$$=2\,(132-143)+3(-99+104)-7(33+32)$$

$$= -22+15+7=0.$$

Thus the lines are concurrent.

By solving simultaneously any two of them, the point of intersection is given by (11, 5).

Exercise 7.6

1. Determine whether the following pairs of lines intersect or are parallel. In case they intersect find the coordinates of their point of intersection.

(*i*) $2x+5y-1=0,\ 3x-y-1=0$
(*ii*) $3x+4y=6,\ 2x+5y=1$
(*iii*) $x-y-1=0,\ 4x=4y+5$
(*iv*) $2x-3y-2=0,\ 6y=4x-1.$

2. Show that the following pairs of lines are coincident :—

(*i*) $x-\frac{y}{2}=1,\quad y=2x-2$

(*ii*) $\frac{x}{3}+\frac{y}{2}=1,\ 2x=6-3y.$

3. Show that the following lines are concurrent. Find also the point of concurrency in each case.

(*i*) $4x+7y-9=0,\ 5x-8y+15=0$ and $9x-y+6=0.$
(*ii*) $2x-y=5,\ x-y=2$ and $11x-5y=28.$
(*iii*) $5x+y+11=0,\ x+7y+9=0$ and $2x+y+5=0.$

ANSWERS

1. (*i*) $\left(\frac{6}{17},\ \frac{1}{17}\right)$
(*ii*) $\left(\frac{26}{7},-\frac{9}{7}\right)$
(*iii*) Parallel
(*iv*) Parallel

3. (*i*) $\left(\frac{-33}{67}, \frac{105}{67}\right)$

(*ii*) (3, 1)

(*iii*) (−2, −1)

7.13 *Equation of a line passing through intersection of two lines.*

Let $l_1 = a_1x + b_1y + c_1 = 0$ (1)

and $l_2 = a_2x + b_2y + c_2 = 0$ (2)

be the two given lines. Then

$$l_1 + kl_2 = 0, \tag{3}$$

where k is a constant, is a straight line as it is linear and passes through the intersection of the given lines, as (3) is satisfied by the coordinates of the point which satisfies both (1) and (2).

As k is at our choice the line (3) can be made to satisfy one given condition.

Example. Find the equation of a line passing through the point (1, 1) and the point of intersection of

$$2x - y = 0 \text{ and } 3x + 2y - 1 = 0.$$

Solution. Line through the intersection of the given lines is

$$(2x - y) + k(3x + 2y - 1) = 0.$$

It passes through (1, 1) if

$$(2-1) + k\,(3+2-1) = 0$$

$$\text{or } k = -\tfrac{1}{4}$$

∴ the equation of required line is

$$(2x - y) - \tfrac{1}{4}\,(3x + 2y - 1) = 0$$

$$\text{or} \quad 5x - 6y + 1 = 0.$$

Exercise 7.7.

1. Find the equation of a line through the origin and the point of intersection of

$$x-2y-1=0$$

and $$2x+y+1=0.$$

2. Find the equation of a line parallel to x-axis through the point of intersection of

$$3x-y-1=0$$

and $$x+2y+2=0.$$

3. Find the equation of a line parallelto $x-y+1=0$ and through the point of intersection of

$$x+3y-1=0$$

and $$3x-y=0.$$

4. Find the equation of a line perpendicular to $3x+y-1=0$ and through the point of intetsection of $2x-y=0$ and $3x+2y=0$.

ANSWERS

1. $3x-y=0$
2. $y+1=0$
3. $5x-5y+1=0$
4. $x-3y=0$

7.14. *Angle from one line to another in terms of their slopes.*

Let l_1 and l_2 be two lines whose inclinations are α_1 and α_2 respectively.

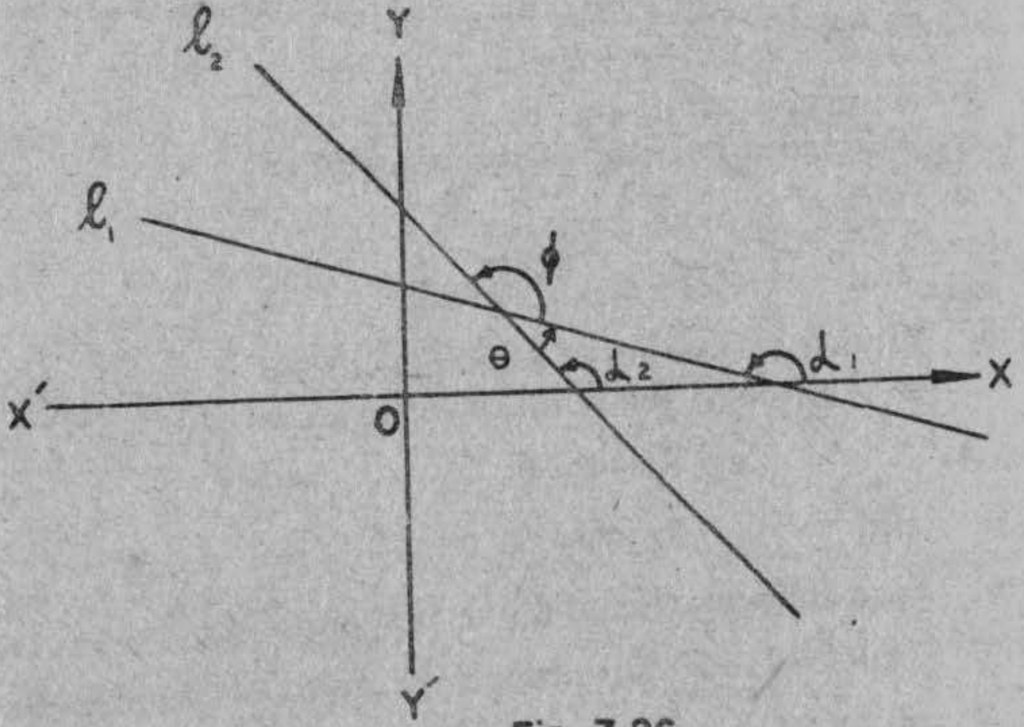


Fig. 7.26

For definiteness let $\alpha_1 > \alpha_2$.

Let Tan $\alpha_1 = m_1$, Tan $\alpha_2 = m_2$.

If θ be the angle from l_2 to l_1 then $\theta = \alpha_1 - \alpha_2$.

$$\text{Tan } \theta = \text{Tan } (\alpha_1 - \alpha_2) = \frac{\text{Tan } \alpha_1 - \text{Tan } \alpha_2}{1 + \text{Tan } \alpha_1 \text{ Tan } \alpha_2} = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

From Figure 7.26 the angle $\varnothing$ from l_1 to l_2 $- 180^\circ - \theta$.

$$\text{Tan } \varnothing = \text{Tan } (180^\circ - \theta) = -\text{Tan } \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

We can deal with the case $\alpha_1 < \alpha_2$ in the same manner.

Note : If the lines are parallel, then $\theta = 0$ or π,

so that $\text{Tan}\ \theta = 0$ and hence $m_1 = m_2$.

If the lines are perpendicular then

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, so that $\text{Tan}\ \theta = \infty$ and hence

$1 + m_1 m_2 = 0$ or $m_1 m_2 = -1$.

Example. Find the angles of the triangle with vertices $A(-2, 4)$, $B(4, -1)$ and $C(8, 2)$.

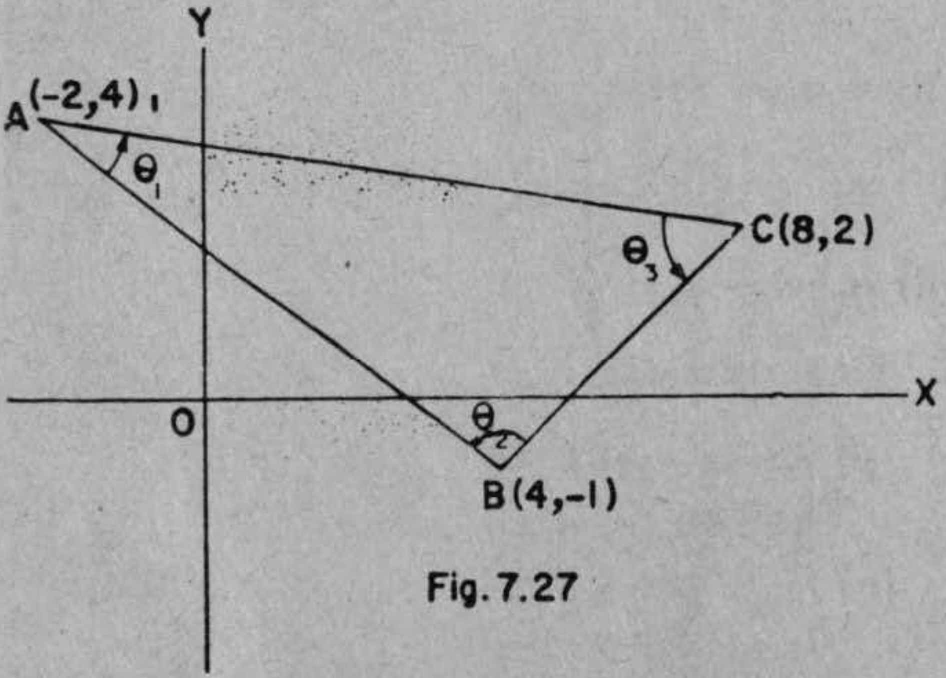


Fig. 7.27

Solution. Slope of line l_1 through A and C is $m_1 = -\frac{1}{5}$

Slope of line through A and B is $m_2 = -\frac{5}{6}$.

Slope of line through B and C is $m_3 = \frac{3}{4}$.

The angle θ_1 at vertex A is formed by going from l_2 to l_1 (remember, the rotation must be counter clockwise) and therefore,

$$\text{Tan } \theta_1 = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{1}{5} - (-\frac{5}{6})}{1 + (-\frac{1}{5})(-\frac{5}{6})} = \frac{19}{35}$$
$$= .5429.$$

From tables we find $\theta_1 = 28° \, 30'$

Similarly, $\text{Tan}\theta_2 = \dfrac{m_2 - m_3}{1 + m_2 m_3} = -4.2222$

and $\text{Tan}\theta_3 = \dfrac{m_3 - m_1}{1 + m_3 m_1} = 1.1176.$

Hence $\theta_2 = 109° \, 19'$ and $\theta_3 = 48° \, 11'$.

Exercise 7.8

1. Find the angle from l_2 to l_1 when their slopes are :—

(i) $m_1 = \sqrt{3}$, $m_2 = \dfrac{1}{\sqrt{3}}$

(ii) $m_1 = \dfrac{1}{\sqrt{3}}$, $m_2 = 1$

(iii) $m_1 = 1, -m_2 = \dfrac{4}{3}$

2. Find the angle from l_2 to l_1 when their equations are :—

(i) $l_1 \equiv 3x - y + 1 = 0,$
$l_2 \equiv 5x + 2y - 1 = 0$

(ii) $l_1 \equiv \sqrt{3}x - y - 1 = 0$
$l_2 \equiv \sqrt{3}x + y - 6 = 0$

(iii) $l_1 \equiv x + 3y + 1 = 0$
$l_2 \equiv 2x + y - 6 = 0.$

3. Find the angles of the triangles ABC whose vertices are :—

(i) A (−3 0). B (2, 5). C (5, 1)
(ii) A (−4 4). B (2, −1), C (10. 3).

4. **The line l_1 passes through the points $(-2, 4)$ and $(5, -1)$. Find the slope of l_2 if angle from l_2 to l_1 is 45°.**

5. **The angle from l_2 to l_1 is 30°. The equation of l_1 is $\sqrt{3}x-y+1=0$. Find the equation of l_2 if it contains $(-3, 1)$.**

ANSWERS

1. (*i*) 30°
   (*ii*) 105°
   (*iii*) $\operatorname{Tan}^{-1}(-7)$.
2. (*i*) $\operatorname{Tan}^{-1}\left(-\frac{11}{13}\right)$
   (*ii*) 120°
   (*iii*) $\operatorname{Tan}^{-1}(7)$.
3. (*i*) $A=37°\ 53'$,
   $B=81°\ 52'$,
   $C=60°\ 15'$,
   (*ii*) $A=53°\ 50'$
   $B=94°\ 24'$,
   $C=31°\ 46'$,
4. $m=-6$.
5. $x-\sqrt{3}\,y+(3+\sqrt{3})=0$.

7.15. *Inequalities of the first degree.*—It is clear that a line divides the plane in which it lies into two regions such that every point of the plane not on the line lies in one of these regions. Thus, if a line is not parallel to the y-axis, every point of the plane not on the line is either above the line or below the line.

We know that the points on the line are characterized by the fact that their coordinates satisfy the equation of the line. The next theorem will show now to characterize the points above (or below) the line in terms of an inequality satisfied by their coordinates.

7.15.1. *Theorem.* Let l denote the line

$$Ax+By+C=0$$

with $B>0$.

(*i*) If P (x_1, y_1) is a point which is above the line l then, $Ax_1+By_1+C>0$.

(*ii*) If P (x_1, y_1) is a point which is below the line l, then, $Ax_1+By_1+C<0$.

Proof: Draw a line k through P (x_1, y_1) parallel to the y-axis intersecting the line l in the point R. Now the x-coordinate of R is the same as the x-coordinate of P, namely x_1. Let us denote the y-coordinate of R by y_2, so

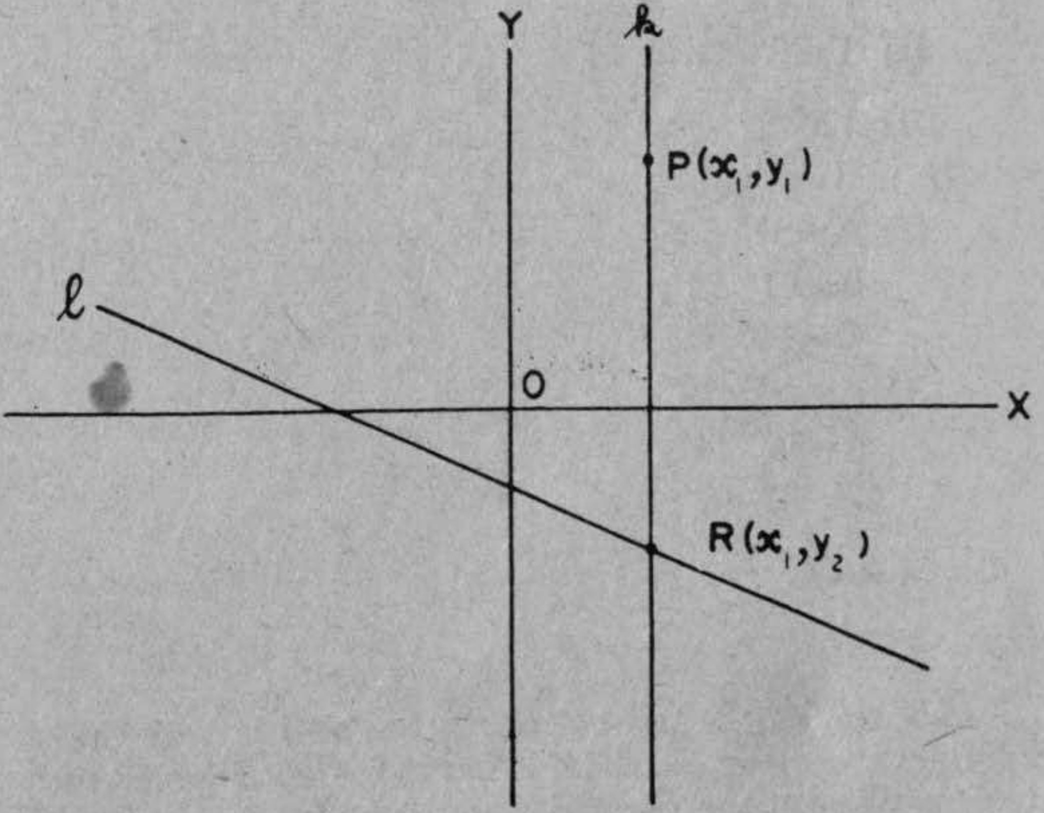


Fig. 7.28

that R is the point (x_1, y_2). Suppose the point P is above l. Then $y_1 > y_2$. Now it is given that $B > 0$. Therefore

$$By_1 > By_2.$$

Adding $Ax_1 + C$ to both sides, we have

$$Ax_1 + By_1 + C > Ax_1 + By_2 + C.$$

Since R (x_1, y_2) is on the line l, $Ax_1 + by_2 + C = 0$

Thus $Ax_1 + by_1 + C > 0$.

If the point P (x_1, y_1) is below the line l, then $y_1 < y_2$ and the proof in this case follows by a simple modification of the above.

Converses of (*i*) and (*ii*) are also true. We state them in the following theorem.

7.15.2. *Theorem.* Let l denote the line

$$Ax + By + C = 0$$

with $B > 0$.

(*i*) If x_1 and y_1 are real numbers such that

$$Ax_1 + By_1 + C > 0$$

then the point P (x_1, y_1) is above the line l.

(*ii*) If x_1 and y_1 are real numbers such that

$$Ax_1 + By_1 + C < 0$$

then the point P (x_1, y_1) is below the line l.

Proof : Since $Ax_1 + By_1 + C > 0$, P (x_1, y_1) cannot be on the line l. Suppose P (x_1, y_1) is below the line. Then by Theorem 7.15.1, $Ax_1 + By_1 + C < 0$, which is a contradiction. Hence P (x_1, y_1) is above the line l.

(*ii*) is proved in a similar way.

Note : We can use Theorem 7.15.1 and Theorem 7.15.2 only after the equation of the given line has been put in the form $Ax + By + C = 0$ with $B > 0$. Of course, this is impossible in case $B = 0$, but then the line l would be parallel to the y-axis and it makes no sense to talk about a point being above or below l (See problem 6 of Exercise 7.9).

Example 1. Find whether each of the points (10, 18), (−12, −14), (50, 80) and (−100, −152) is above or below the line whose equation is $3x-2y+7=0$.

Solution. We rewrite the equation in the foom

$$-3x+2y-7=0.$$

Since, $-3(10)+2(18)-7=-1<0$, the point (10, 18) is below the line l;

$-3(-12)+2(-14)-7=1>0$, the point (−12, −14) above the line l;

$-3(50)+2(80)-7=3>0$, the point (50, 80) is above the line l;

$-3(-100)+2(-152)-7=-11<0$, the point (−100, −152) is below the line l.

Exercise 7.9

In the following questions determine whether each of the specified points is above or below the given line.

1. $3x+11y-44=0$; (10, 1), (−4, 6), (5, 3).
2. $10x-12y+17=0$: (5, 5), (−20, −15), (100, 84).
3. $5x+14y-11=0$; (11, −3), (−4, 2), (20, −7).
4. $x+2y-12=0$; (7, 3), (14, −1), (−8, 11).
5. Show that the points (101, 50) lies between the parallel lines

$$x-2y+1=0$$
$$2x-4y-3=0$$

6. Let l denote the line $Ax+C=0$, with $A>0$. Show that if $P(x_1, y_1)$ is a point which is to the right of l, then $Ax_1+C>0$ whereas if $P(x_1, y_1)$ is to the left of l, then $Ax_1+C<0$.

ANSWERS

1. Below, above, above.
2. Below, above, below.

3. Above, below, below.

4. Above, on, above.

7.16. *Distance of a point from a line*—If l is a given line (Figure 7.29) and P is a given point, let us denote by Q the projection of P on l.

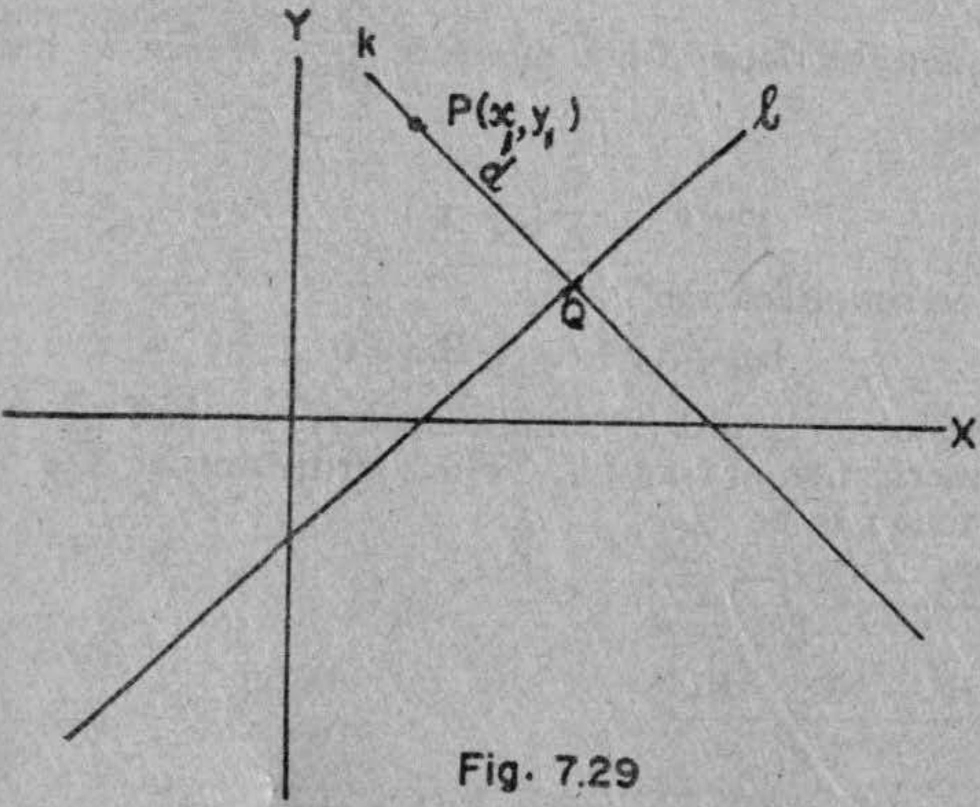


Fig. 7.29

By the distance of the point P from the line l, we shall mean $|PQ|$; in other words, the distance is to be measured along a perpendicular to the line l. We now prove the following Theorem.

7.16.1. *Theorem.* The distance of the point P (x_1, y_1) from the line

$$Ax+By+C=0 \qquad \ldots\ldots(1)$$

is given by the formula

$$\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}.$$

Proof: We shall first find the equation of the line k through P (x_1, y_1) and perpendicular to l.

If l is not parallel to a coordinate axis, its slope is $-\frac{A}{B}$, and the slope of k is therefore $\frac{B}{A}$. Hence k has its equation

$$y-y_1=\frac{B}{A}(x-x_1)$$

which on simplification is

$$Bx-Ay+Ay_1-Bx_1=0 \qquad \ldots\ldots(2)$$

Now we solve (1) and (2) simultaneously to find the point Q of intersection of l and k. The coordinates of the point Q are found to be

$$x=\frac{B^2x_1-ABx_1-AC}{A^2+B^2}$$

$$y=\frac{A^2y_1-ABx_1-BC}{A^2+B^2}$$

Now the distance between P and Q is given by

$$d^2=\left(\frac{B^2x_1-ABy_1-AC}{A^2+B^2}-x_1\right)^2+\left(\frac{A^2y_1-ABx_1-BC}{A^2+B^2}-y_1\right)^2$$

$$=\left[\frac{A(Ax_1+By_1+C)}{A^2+B^2}\right]^2+\left[\frac{B(Ax_1+By_1+C)}{A^2+B^2}\right]^2$$

$$=(A^2+B^2)\left(\frac{Ax_1+By_1+C}{A^2+B^2}\right)^2=\frac{(Ax_1+By_1+C)^2}{A^2+B^2}.$$

Hence $d=\dfrac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$.

It is easy to verify that the formula holds in the case when the line is parallel to either of the axes.

Example. Find the distance of the point $(-3, -2)$ from the line $4x-5y-7=0$

Solution. $d=\dfrac{|4(-3)-5(-2)-7|}{\sqrt{4^2+(-5)^2}}=\dfrac{|-9|}{\sqrt{41}}$

or $d=\dfrac{9\sqrt{41}}{41}$.

7.17. *Area of a triangle*. Find the area of a triangle whose vertices are given.

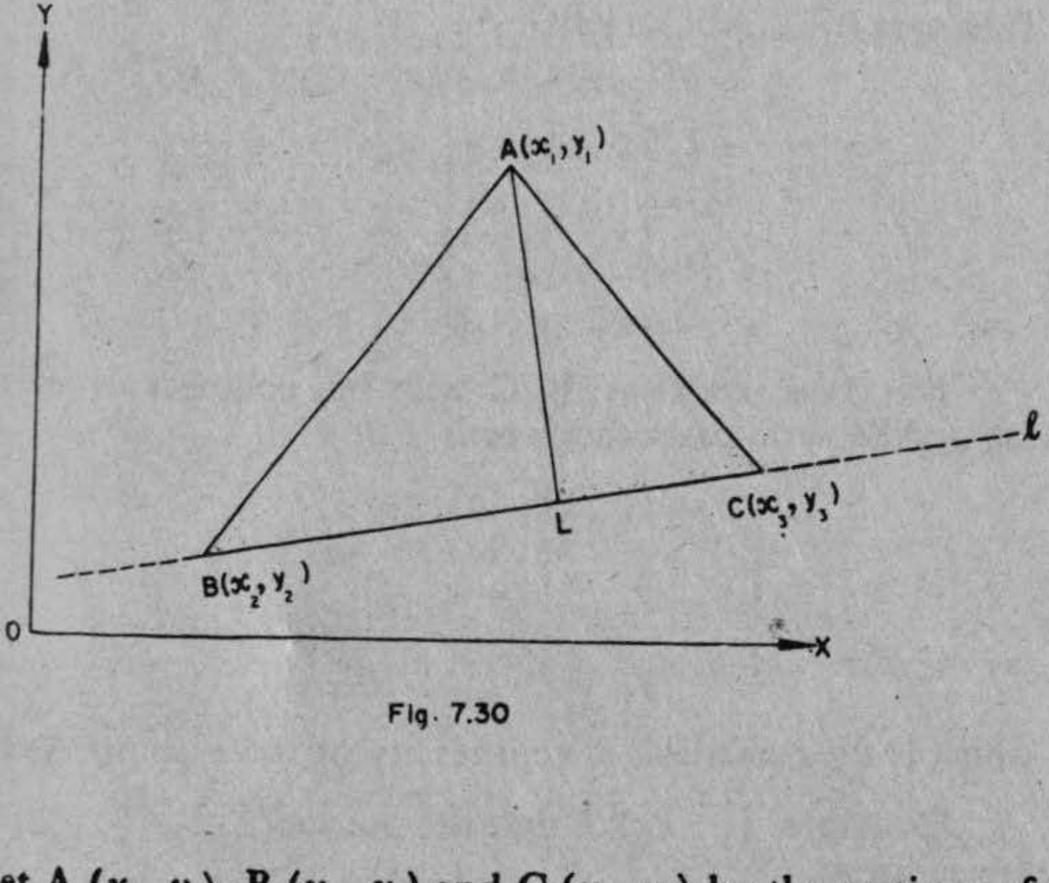


Fig. 7.30

Let A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) be the vertices of a given triangle ABC. Draw AL perpendicular from A to BC.

By elementary geometry, we know that area of triangle ABC$-\frac{1}{2}$ BC. AL.

$$\text{Now BC} = \sqrt{(x_2-x_3)^2+(y_2-y_3)^2}.$$

To find AL we proceed as follows :

Equation of the line l is (BC is a segment of the line l)

$$y-y_2 = \frac{y_3-y_2}{x_3-x_2}(x-x_2)$$

$$\text{or} \quad (y_2-y_3)x+(x_3-x_2)y+x_2y_3-x_3y_2=0.$$

AL = Distance of A from the line l

$$=\frac{(y_2-y_3)x_1+(x_3-x_2)y_1+x_2y_3-x_3y_2}{\sqrt{(y_2-y_3)^2+(x_2-x_3)^2}}.$$

Thus area of $\triangle$ ABC $=\frac{1}{2}$ BC. AL

$$=\tfrac{1}{2}\{(y_2-y_3)x_1+(x_3-x_2)y_1+x_2y_3-x_3y_2\}$$

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The three points A, B, C will be collinear if area of the $\triangle$ ABC = 0. In other words

$$\text{If} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

which is the condition of collinearity of three points A, B, C.

Example 1. Prove that the points (2, 3), (5, 0), (4, 1) are collinear.

Solution. Since $\begin{vmatrix} 2 & 3 & 1 \\ 5 & 0 & 1 \\ 4 & 1 & 1 \end{vmatrix} =2(-1)-3(5-4)+1(5$ $=-2-3+5=0$, the giv points are collinear.

Example 2. Find the area of the triangle whose vertices are P (1, 2), Q (−2, 3) and R (0, 5).

$$\text{Area of } \triangle PQR = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ -2 & 3 & 1 \\ 0 & 5 & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}\{(3-5)+2(2-5)\}$$

$= \tfrac{1}{2}(-8) = 5$ (omitting the negative sign).

Exercise 7.10

1. Find the distance from the given point to the given line in each of the following cases :

(i) $(3, 2)$; $4x-3y+2=0$

(ii) $(3, 7)$; $3x+4y-2=0$

(iii) $(-6, 5)$; $5x-12y+12=0$

(iv) $(3, -1)$; $5x+12y-16=0$.

2. If A (4, 5), B (−3, 2), C (1, −4) are the vertices of a triangle ABC, find the length of the altitude from A to BC.

3. Find the area of the triangle whose vertices are .

(i) A (4,5), B (−3, 2) and C (1, −4)

(ii) P (2, 1), Q (4, 3) and R (2, 5)

(iii) L (4, 12), M (2, 3) and N (3, −6).

4. Find the area of the quadrilateral whose vertices are :

(i) (2, 4), (3, 2), (8, 4), (7, 6)

(ii) (0, 0), (0, 3), (5, 0), (9, 11)

(iii) (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4).

5. The distance from the point $(-2, 3)$ to the line $4x+3y+k=0$ is 3.

Find the value of k.

6. Find the distance between the two parallel lines $4x-3y-12=0$ and $4x-3y+9=0$.

[Hint : Choose any point on one line and find its distance from the other line].

7. Show that the points $(2, 1)$, $(4, -1)$ and $(1, 2)$ are collinear.

8. Show that the points $(a, b+c)$, $(b, c+a)$, $(c, a+b)$ are collinear.

ANSWERS

1. (*i*) $\frac{8}{5}$
   (*ii*) 7
   (*iii*) 6
   (*iv*) 1.

2. $\frac{27}{13}\sqrt{13}$

3. (*i*) 27
   (*ii*) 4
   (*iii*) $\frac{27}{2}$

4. (*i*) 12
   (*ii*) 41
   (*iii*) $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$

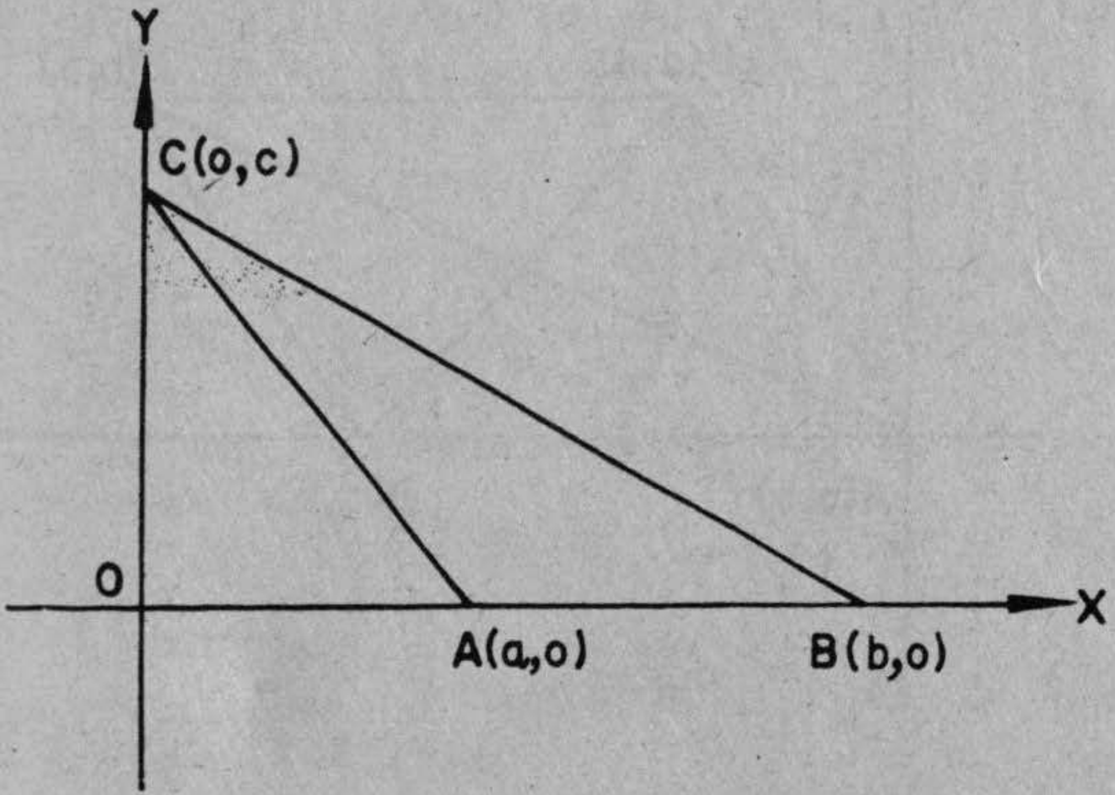
5. 14 and -16

6. $\frac{21}{5}$.

7.18. *Application to Elementary Geometry.* Many problems of Euclidean Geometry can easily be solved by the methods of analytic geomery developed in this chapter. An important observation is that geometrical properties of figures are not affected by the position of the coordinate axes, and a convenient choice of axes may greatly simplify the calculations.

Example 1. Prove that the altitudes of any triangle are concurrent.

Solution. Let ABC be the given triangle.



Fig· 7.31

Let the line AB be taken as x-axis and the straight line through C perpendicular to AB be taken as y-axis. Clearly this choice of axes can be made for any triangle. The coordinates of the vertices are then of the form A $(a, 0)$, B $(b, 0)$, and C $(0, c)$. The altitude of the triangle from C has the equation $x=0$. (*i*)

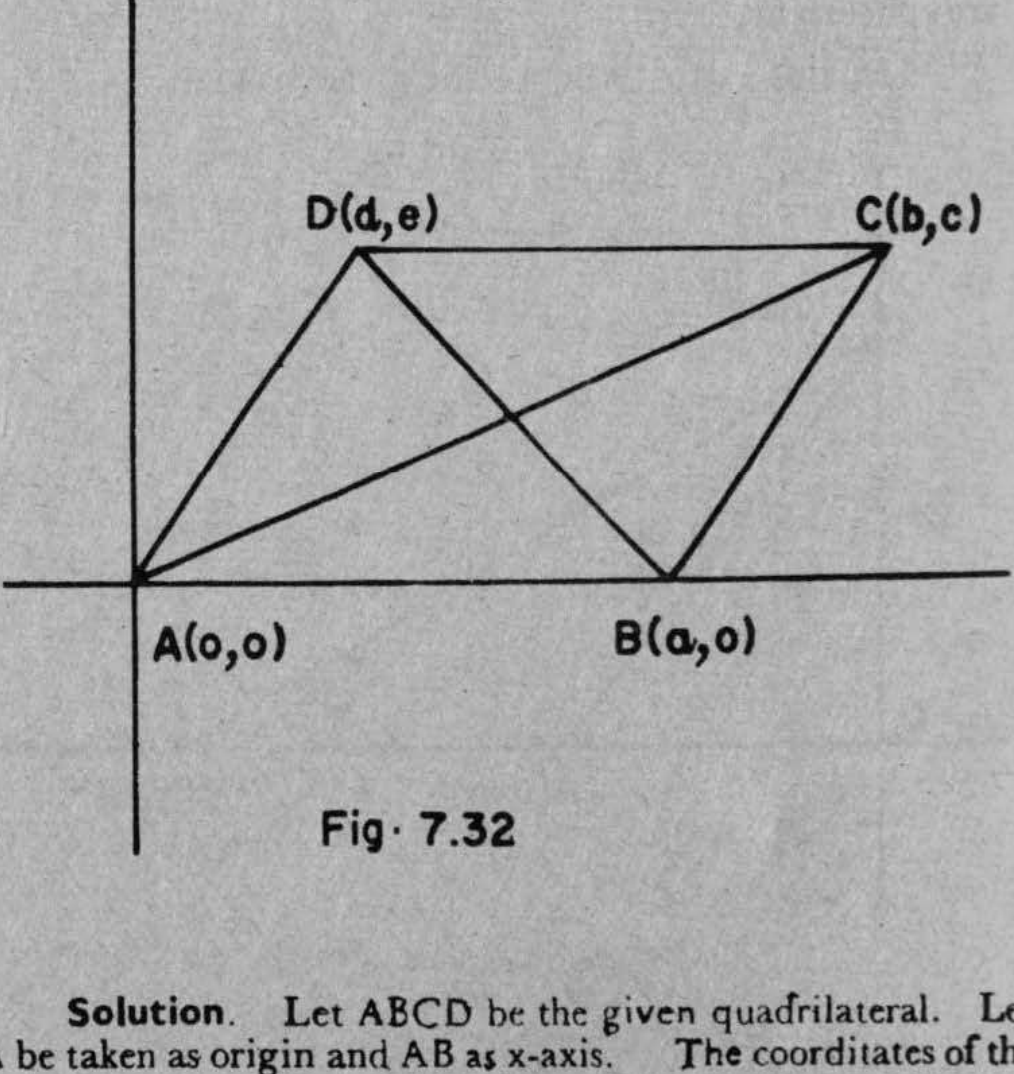
Simple calculations show that the altitude from A is

$$bx--cy-ab=0 \qquad (ii)$$

and from B is $ax-cy-ab=0$ (iii)

Solving equations (*i*) and (*ii*), their common point is $\left(0,\ -\frac{ab}{c}\right)$ which also satisfies equation (*iii*). Hence the assertion.

Example 2. Prove that a quadrilateral whose diagonals bisect each other is a parallelogram.



Fig· 7.32

Solution. Let ABCD be the given quadrilateral. Let A be taken as origin and AB as x-axis. The coorditates of the vertices are then as indicated in figure 7.32. Now $\left(\frac{b}{2},\frac{c}{2}\right)$ is ths mid. point of the diagonal AC and $\left(\frac{a+d}{2}.\ \frac{e}{2}\right)$ is the mid.

point of the diagonal BD. By hypothesis

$$\frac{b}{2}=\frac{a+d}{2} \text{ and } \frac{c}{2}=\frac{e}{2}$$

$$\text{or} \quad b=a+d \text{ and } c=e.$$

Since $c=e$, side CD has slope zero and is, therefore, parallel to side AB. If $d\neq 0$, the slope of side AD is $\frac{e}{d}$ and the slope of BC is $\frac{c}{b-a}$. Now $\frac{c}{b-a}=\frac{e}{d}$ ($\because b=a+d$, $c=e$).

Hence AD // BC.

If $d=0$, then AD and BC are paralled to the y-axis. Thus in every case, the figure is a parallelogram.

Exercise 7.11

1. Prove that an isosceles triangle has two equal medians.

2. Prove that any triangle which has two equal medians is an isosceles triangle.

3. Show that by a proper choice of coordinate axes the vertices of any parallelogram have coordinates of the form $(0, 0)$, $(a, 0)$, (b, c) and $(b-a, c)$.

4. Prove that the diagonals of a parallelogram bisect each other.

5. Prove that the sum of squares of the lengths of the sides of any parallelogram equals the sum of the squares of the length of the diagonals.

6. Prove that the figure obtained by joining in order the mid. points of the sides of any quadrilateral is a parallelogram.

7. Show that the lines joining the mid. points of opposite sides of any quadrilateral intersect at a point which bisects both lines. Show also that this same point bisects the line joining the mid. points of the diagonals of the adrilateral.

8. Prove that in a triangle the sum of the squares of the lengths of the medians is three-fourth the sum of the squares of the lengths of the sides.

9. Prove that the distance between the mid. points of the non-parallel sides of a trapezoid equals half the sum of the lengths of the parallel sides.

10. In any triangle the perpendicular bisectors of the sides meet in a point.

7.19. *Equation of locus*. In § 7.1 we discussed the locus of an equation. In this and the following chapters we derive equations of certain loci each defined by some geometric conditions. We may recall, by the equation of a locus in R^2 is meant an equation which has the given locus as its solution set.

Example 1. Find the equation of the locus of all points which are equidistant from the point A (0, 2) and the point B (3, 1).

Solution. Let P (x, y) be any point on the locus. Then, $| PA | = | PB |$

$$\therefore \quad \sqrt{x^2+(y-2)^2}=\sqrt{(x-3)^2+(y-1)^2}$$

Squaring and simplifying, we obtain the required locus as

$$3x-y-3=0.$$

Example 2. Find the equation of the locus of all points such that the slope of the line through P and the point A (2, 0) is twice the slope of the line through P and the point B $(-2, 0)$.

Solution. Let P (x, y) be any point satisfying the stated condition. Now slope of the line **PA** is $\frac{y}{x-2}$ and the slope of the line PB is $\frac{y}{x+2}$. Hence we have

$$\frac{y}{x-2}=2\,\frac{y}{x+2}$$

as the equation of the given locus.

In this equation $x \neq 2$ or -2, since division by zero is not permissible. Hence we may simplify this equation to obtain $y\,(x-6)=0$ as the required locus.

Thus the given locus consists of lines $y=0$ and $x=6$ with the two points A $(2, 0)$ and B $(-2, 0)$ excluded.

Exercise 7.12

1. Find the equation of the locus of points which are at a distance of 3 units from the point $(-1, 2)$.

2. Find the equation of the locus of points which are equidistant from the points $(-5, 1)$ and $(5, 3)$.

3. Find the equation of the locus of point which are equidistant from the point $(4, 2)$ and the y-axis.

4. Find the equation of the locus of the point whose distance from the point $(-1. 0)$ is double its distance from the point $(2, 0)$.

5. Find the equation of the locus of points which are equidistant from the two parallel lines $3x-2y+4=0$ and $3x-2y-8=0$.

6. Find the equation of the locus of the point P such that the slope of the line through P and the origin is equal to one half the slope of the line through P and the point $(1, -2)$.

7. Find the equation of the locus of points which are equidistant from the point $(1, -1)$ and from the line $x+y-2=0$.

ANSWERS

1. $x^2+y^2+2x-4y-4=0$
2. $5x+y-2=0$
3. $y^2-4y-8x+20=0$
4. $x^2+y^2-6x+5=0$
5. $3x-2y-2=0$
6. $xy=2(x+y)$
7. $x^2+y^2+8y-2xy=0.$

Miscellaneous Exercise 3

1. Two opposite vertices of a square are $(-2, 6)$ and $(5, 5)$. Find the coordinates of the other two vertices.

2. Show that for all values of θ, the distance between the points (x, y) and $(x+\mu \text{ Cos } \theta, y+\mu \text{ Sin } \theta)$ is μ.

3. In what ratio is the line segment joining the points $(1, 2)$ and $(-3, -6)$ divided by

(*i*) x-axis

(*ii*) y-axis.

4. Show that the medians of a triangle are concurrent. Find also the point of concurrency.

5. If D is the mid point of the side BC of a $\triangle$ ABC, show that $AB^2+AC^2=2(AD^2+BD^2)$.

6. Prove that the mid. point of the hypotenuse of an isosceles right triangle is equidistant from the vertices.

7. Show that the line segment joining the mid. points of two sides of a triangle is parallel to the third side and equal to one-half its length.

8 Show that the $\triangle$ ABC with vertice A $(1, -3)$, B $(3, -1)$, C $(4, -2)$ is a right triangle.

9. Discuss the following for intercepts and symmetry :

(*i*) $(x+1)y^2=x$

(*ii*) $y^2\,(x-3)^2=1$

(*iii*) $y\,(x-2)^2+2\,(x^2-1)$.

10. Find the equation of the line through the mid. point of AB, where A, B are $(-3, 1)$, $(2, -1)$ respectively and parallel to the line through C $(1, 2)$, D $(-4, 1)$.

11. Show that the following points are collinear :

(*i*) $(-1, -1)$, $(5, 7)$, $(8, 11)$

(*ii*) $(a, b+c)$, $(b, c+a)$, $(c, a+b)$.

Hint : The area of the triangle (whose verties are these points) is zero.

12. Find the equation of a straight line in the form $\frac{x-x_1}{\text{Cos}\,\theta}=\frac{y-y_1}{\text{Sin}\,\theta}=r$, where r is the distance of a point (x, y) from the point (x_1, y_1) on it and θ is the angle which the line makes with x-axis.

13. Find the ratio in which the join of $(-3a, b)$ and $(2a, 5\,b)$ is divided by the line $\frac{x}{a}+\frac{y}{b}=1$.

14. A line is drawn through P $(1, 2)$ inclined at an angle of $\frac{\pi}{4}$ to the x-axis. Find the coordinates of two points on it, distant 2 units from P on either side of it.

15. Find the condition that the three lines

$$\begin{aligned}&ax+hy+g=0\\&hx+by+f=0\\&gx+fy+c=0\end{aligned}$$

intersect in one point.

16. State the geometrical meanings of the constants in the equations.

(*i*) $y=mx+c$

(*ii*) $\frac{x}{a}+\frac{y}{b}=1.$

(*iii*) $x \text{ Cos } \alpha+y \text{ Sin } \alpha=p.$

ANSWERS

1. (1, 2), (2, 9)

3. (*i*) 1 : 4

(*ii*) 1 : 3

9. (*i*) Intercept (0, 0); Domain : All real numbers except $-1\leqslant x<0$; Range : all non-negative values of y except $y=1$. Symmetry about x-axis.

(*ii*) Intercepts $(0, \frac{1}{3})$, $(0, -\frac{1}{3})$; Symmetry about x-axis.

Domain : all reals except $x=3$

Range : all reals except $y=0$.

(*iii*) Intercepts : (0, 1), (1, 0), (—1, 0) ;

Domain : all reals except $x=\pm\sqrt{2}$

Range : all reals except $y=2$.

Symmetry w.r.t. y-axis.

10. $2x-10y+1=0$

13. 1 : 2

14. $(\sqrt{2}+1, \sqrt{2}+2), (1-\sqrt{2}, 2-\sqrt{2})$

15. $abc+2fgh-af^2-bg^2-ch^2=0$

CHAPTER VIII

CONIC SECTIONS

8.1. *Introduction.* In this chapter we study the properties of the *circle*, the *parabola*, the *ellipse* and the *hyperbola*. Each of these curves is described geometrically and is the graph of a quadratic equation in x, y. The parabola, ellipse and hyperbola are called *conic sections* (or simply *conics*). The circle is a special case of an ellipse.

8.2. *Equation of a circle.* A circle is the set of points whose distance from a given fixed point is constant. The fixed point is called the *centre* of the circle and the constant distance is called the *radius* of the circle.

Let C (h, k) be the centre and r the radius of the circle.

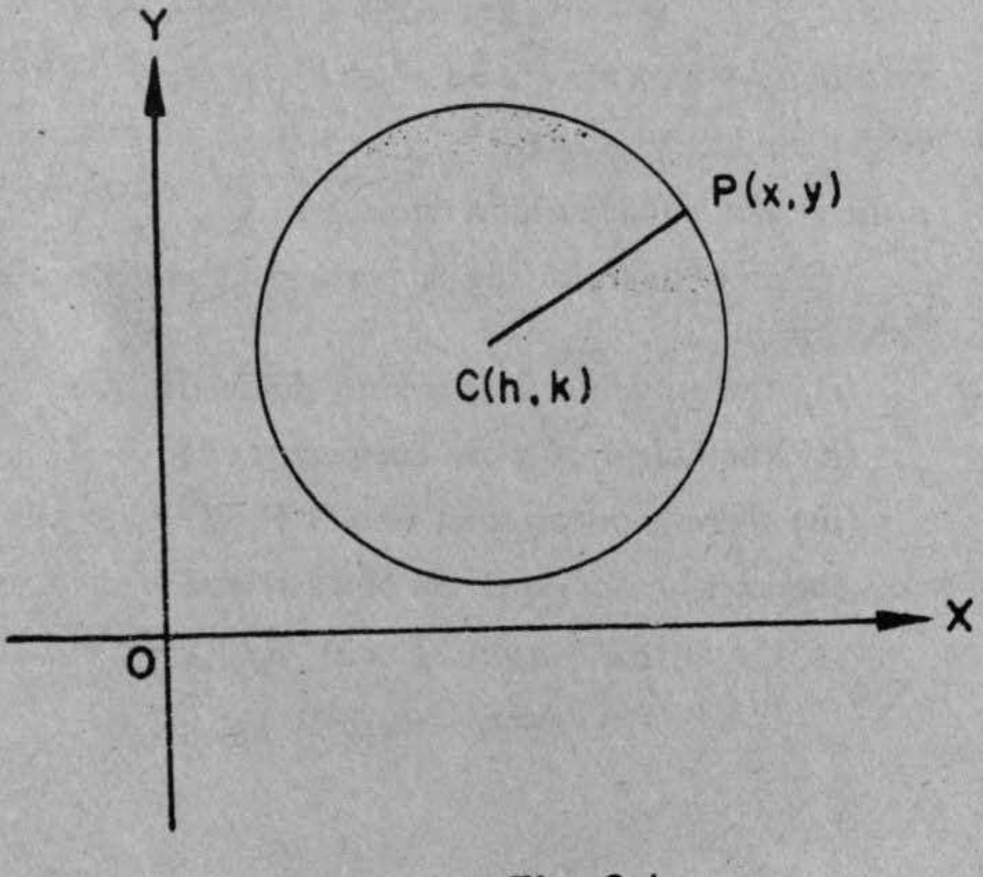


Fig. 8.1

If P (x, y) is any point on the circle, then

$$|CP| = r \text{ and hence}$$

$$\sqrt{(x-h)^2+(y-k)^2}=r.$$

Thus $(x-h)^2+(y-k)^2=r^2$ $\qquad$ ……… (1)

s the equation of the circle with centre (h, k) and radius r.

If the centre is at the origin, then $(h, k) = (0,0)$ and quation (1) becomes

$$x^2+y^2=r^2 \qquad \text{………(2)}$$

Example 1. Find the equation of the circle whose entre is $(3, -1)$ and radius 6.

Solution. Required equation is

$$(x-3)^2+(y+1)^2=6^2$$

$$\text{or } x^2-6x+9+y^2+2y+1=36$$

$$\text{or } x^2+y^2-6x+2y-26=0.$$

8.3. *The General Equation of a Circle.* The equation $x-h)^2+(y-k)^2=r^2$ reduces on simplification to

$$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0.$$

Putting $h=-g$, $k=-f$ and $h^2+k^2-r^2=c$

ve get $\quad x^2+y^2+2gx+2fy+c=0$

is the *general* equation of a circle.

We note the following properties of the general equation of a circle.

(*i*) the equation is of second degree in x, y,

(*ii*) coefficient of x^2 = coefficient of y^2 = 1,

(*iii*) there is no product term xy.

The *most general* equation of a circle is

$$ax^2+ay^2+2g'x+2f'y+c'=0.$$

This form of the equation of a circle has all the properties (*i*), (*ii*) and (*iii*) except that the coefficient of x^2 and y^2 is 'a' and not 1 as in (*ii*) above. To obviate this difficulty we divide this equation throughout by 'a' and obtain

$$x^2+y^2+2\frac{g'}{a}x+2\frac{f'}{a}y+\frac{c'}{a}=0.$$

$$\text{Writing}\ \frac{g'}{a}=g,\ \frac{f'}{a}=f \text{ and } \frac{c'}{a}=c,$$

the last equation reduces to

$$x^2+y^2+2gx+2fy+c=0,$$

which is the general equation of a circle.

8.4. Show that the equation

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots(i)$$

always represents a circle. Also find its centre and radius.

The equation (*i*), after completing the squares, can be written as

$$(x+g)^2+(y+f)^2=g^2+f^2-c.$$

Comparing this with equation (1) of § 8.2 we see that equation (*i*) represents a circle whose centre is the point $(-g, -f)$ and whose radius is $\sqrt{g^2+f^2-c}$.

Example 1. Find the centre and radius of the circle,

$$x^2+y^2-4x+6y-12=0$$

and draw the graph.

Solution. Here $g=-2$, $f=3$, $c=-12$ and $g^2+f^2-c=4+9+12=25$. Thus the centre of the given circle is $(2, -3)$ and radius $=5$.

The graph of the circle is shown in Figure 8.2.

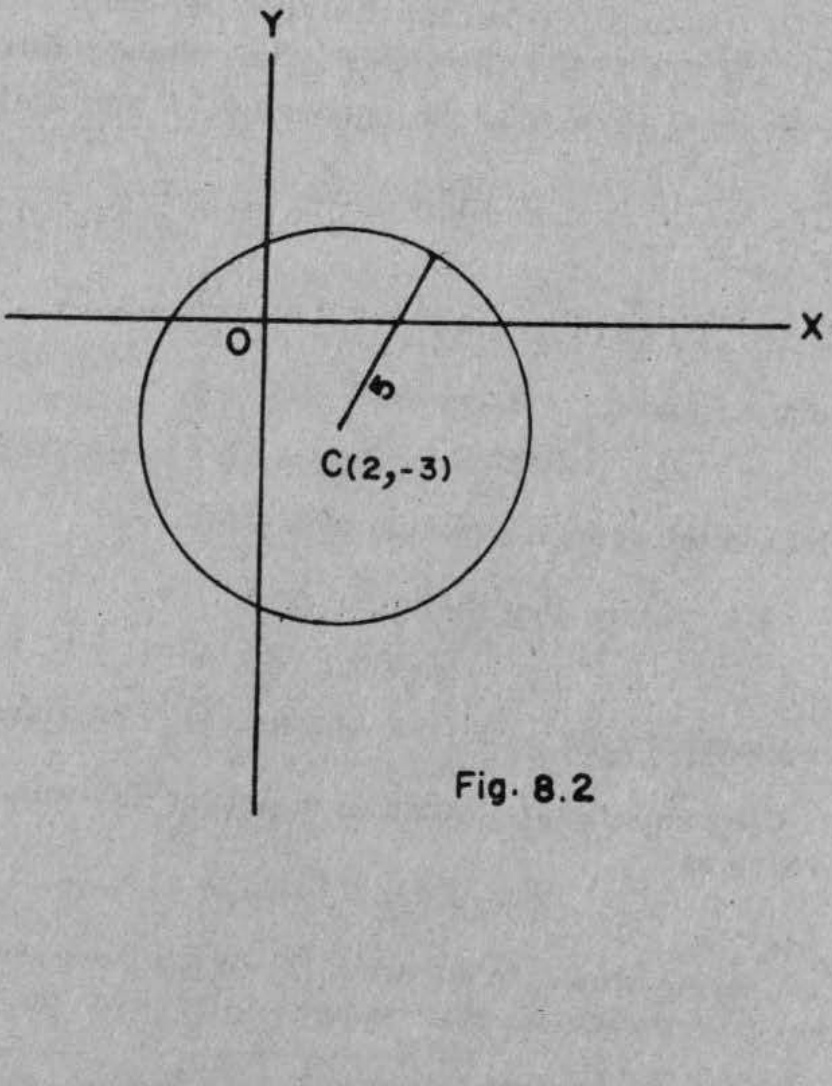


Fig. 8.2

Example 2. Find the centre and radius of the circle

$$4x^2 + 4y^2 + 24x - 8y + 15 = 0.$$

Solution. Dividing the equation by 4 we have,

$x^2 + y^2 + 6x - 2y + \frac{15}{4} = 0$, which is the general equation. Rearranging we have

$$(x^2 + 6x) + (y^2 - 2y) = -\tfrac{15}{4}$$

$$\text{or } (x^2 + 6x + 9) + (y^2 - 2y + 1) = -\tfrac{15}{4} + 9 + 1$$

or $$(x + 3)^2 + (y - 1)^2 = \tfrac{25}{4}.$$

Thus the centre is C $(-3, 1)$ and the radius is $\frac{5}{2}$.

Exercise 8.1.

1. Find the equation of each of the following circles :—

   (*i*) Centre at $(0, 0)$; radius 5,

   (*ii*) Centre at $(2, 4)$; radius 5,

   (*iii*) Centre at $(9, -3)$; radius 3,

   (*iv*) Centre at $(\frac{7}{4}, -\frac{9}{2})$; radius $\frac{1}{2}$.

2. Find the equation of the circle having $(-5, 6)$ and $(1, -2)$ as the end points of a diameter.

3 Find the equation of the circle having the centre at the origin and containing the point $(4, -3)$.

4. Show that the equations $x = r\cos\theta$ and $y = r\sin\theta$ represent a circle with its centre at the origin and radius equal to r.

5. Find the centre and the radius of each of the following circles and draw the figures :—

   (*i*) $x^2 + y^2 - 8x + 10y + 5 = 0$

   (*ii*) $x^2 + y^2 - 4x + 6y - 36 = 0$

   (*iii*) $x^2 + y^2 + 12x - 10y - 20 = 0$

   (*iv*) $4x^2 + 4y^2 - 12x + 4y - 15 = 0$

   (*v*) $3x^2 + 3y^2 - 4x + 10y - 7 = 0$

   (*vi*) $x^2 + y^2 - 4x - 6y + 13 = 0$

   (*vi*) $x^2 + y^2 - 4x - 6y + 13 = 0$

   (*vii*) $x^2 + y^2 - 6x + 2y + 15 = 0$

ANSWERS

1. (*i*) $x^2 + y^2 - 25 = 0$
   (*ii*) $x^2 + y^2 - 4x - 8y - 5 = 0$
   (*iii*) $x^2 + y^2 - 18x + 6y + 81 = 0$
   (*iv*) $16x^2 + 16y^2 - 56x + 144y + 369 = 0$

2. $x^2 + y^2 + 4x - 4y - 17 = 0$

3. $x^2 + y^2 - 25 = 0$

5 (*i*) Centre $(4, -5)$; radius 6
   (*ii*) Centre $(2, -3)$; radius 7
   (*iii*) Centre $(-6, 5)$; radius 9
   (*iv*) Centre $(\frac{3}{2}, -\frac{1}{2})$; radius $\frac{5}{2}$
   (*v*) Centre $(\frac{2}{3}, -\frac{5}{3})$; radius $\frac{5}{3}\sqrt{2}$
   (*vi*) Centre $(2, 3)$; radius 0
   (*vii*) Not a real circle.

8.5. *Circles Determined by Given Conditions.* We have already established that the general equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \cdots(1).$$

Since there are three independent constants g, f and c in (1), any three geometric conditions which lead to three independent equations in g, f and c will determine a circle. These three equations may be solved simultaneously, and the resulting values of g, f and c when substituted in (1) determine the equation of the circle.

Example 1. Find the equation of the circle which contains the points $(3, -3)$, $(0, -1)$ and $(3, -1)$.

Solution. The general form of the equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

The coordinates of each of the points which belong to the circle must satisfy this equation.

Hence
$$\begin{aligned} 9+9+6g-6f+c&=0\\ 0+1\qquad\;\; -2f+c&=0\\ 9+1+6g-2f+c&=0. \end{aligned}$$

Solving these three equations simultaneously we have
$$g=-\tfrac{3}{2},\ f=2 \text{ and } c=3.$$

Thus the equation is
$$x^2+y^2-3x+4y+3=0.$$

Example 2. Find the equation of the circle which contains the point $(0, 1)$ and touches the line $x+2y+2=0$ at the point $(4, -3)$.

Solution. Let (h, k) be the centre of the given circle. Since the points $(0, 1)$ and $(4, -3)$ belong to the circle we must have
$$h^2+(k-1)^2=(h-4)^2+(k+3)^2$$
$$\text{i. e.}\quad h-k-3=0 \qquad (1).$$

We know from elementary geometry that a tangent to a circle is perpendicular to the radius through the point of contact. Hence the radius through $(4, -3)$ is perpendicular to the given tangent $x+2y+2=0$, which is a line of slope $-\frac{1}{2}$.

Thus the slope of the radius must be 2.

Hence
$$\frac{k+3}{h-4}=2$$
$$\text{i. e.}\quad 2h-k-11=0 \qquad (2).$$

Solving (1) and (2) simultaneously, we get $h=8$, $k=5$. Thus the centre of the required circle is $(8, 5)$.

The radius of the required circle is the distance between $(8, 5)$ and one of the known points, say $(0, 1)$, on the circle. The radius is found to be $\sqrt{80}$. Thus the required equation of the circle is
$$(x-8)^2+(y-5)^2=80.$$

Example 3. Find the equation of the circle which touches the line $3x + 4y + 4 = 0$ and which contains the points $(0, 0)$ and $(4, 0)$.

Solution. Suppose that (h, k) is the centre of the required circle. Then (h, k) is equidistant from $(0, 0)$ and (4.0). Hence $h^2 + k^2 = (h - 4)^2 + k^2$ or $h = 2$.

Since the required circle touches the line $3x + 4y + 4 = 0$, the distance of the centre $(2, k)$ from this line must be equal to the radius of the circle and, therefore, equal to the distance between $(0, 0)$ and the centre $(2, k)$.

$$\text{Hence} \quad \frac{|3 \cdot 2 + 4 \cdot k + 4|}{5} = \sqrt{4 + k^2}$$

$$\text{or} \quad |10 + 4k| = 5\sqrt{4 + k^2}.$$

Squaring, we have

$$9k^a - 80k = 0.$$

This equation has solution set $\left\{0, \frac{80}{9}\right\}$.

Thus there are two circles which satisfy the given conditions, one with centre $(2, 0)$ and the other with centre $(2, \frac{80}{9})$. The radius of the first circle is 2 and that of the second is $\frac{88}{9}$ (how ?).

Hence the required circles are

$$(x - 2)^2 + y^2 = 4.$$

$$\text{and } (x - 2)^2 + \left(y - \frac{80}{9}\right)^2 = \frac{6724}{81}$$

Exercise 8.2

1. Find the equation of the circle which contains each of the following sets of three points :—

(*i*) $(0, 0)$, $(4, 4)$ and $(8, 0)$

(*ii*) $(2. - 4)$, $(4, 0)$ and $(7, 1)$

(*iii*) $(-1, 0), (-1, -4)$ and $(3, -4)$

(*iv*) $(1, 4), (2, -3)$ and $(3, 0)$.

2. Find the equation of the circle whose centre is at $(-3, 5)$ and which touches the line $4x - 3y - 18 = 0$.

3. Find the equation of the circle which touches both the axes and is of radius 3

4. Find the equation of the circle containing the point $(-7, 3)$ and touching the line $x - 2y + 3 = 0$ at the point $(-1, 1)$.

5 Find the equation of the circle containing the point $(6, 0)$ and touching the line $y = x$ at the point 4, 4).

6. Find the equation of the circle containing the points $(-1, -1)$ and $(3, 1)$ and with centre on the line

$$x - y + 10 = 0.$$

7. Find the equation of the circle containing the points $(-1, -2)$ and $(6, -1)$ and touching x — axis.

8. Find the equation of the circle concentric with the circle $x^2 + y^2 + 6x - 10y + 33 = 0$ and touching the x — axis.

9. Find the equation of the circles with centres at $(1, -7)$ and touching the circle with the equation

$$x^2 + y^2 + 4x - 2y + 1 = 0.$$

ANSWERS

1. (*i*) $x^2+y^2-8x=0$

(*ii*) $x^2+y^2-14x+8y+40=0$

(*iii*) $x^2+y^2-2x+4y-3=0$

(*iv*) $x^2+y^2+4x-21=0$.

2. $x^2+y^2+6x-10y-47=0$.

3. Equation of the circle in the first quadrant is $(x-3)^2+(y-3)^2=9$. Find equations if the circle lies in other quadrants.

4. $x^2+y^2+6x-10y+14=0$

5. $3(x^2+y^2)-34x-14y+96=0$

6 $3x^2+3y^2+16x-44y-34=0$

7. $(x-3)^2+(y+5)^2=25$, $(x-23)^2+(y+145)^2=21025$.

8 $x^2+y^2+6x-10y+9=0$

9. $x^2+y^2-8x+14y+1=0$, $x^2+y^2-8x+14y-79=0$.

8.6. *Circles through the points of intersection of two given circles.* Let the equations of the two given circles be

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad \ldots\ldots(1)$$

and $$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad \ldots\ldots(2)$$

Let k be a real number and consider the equation

$$\begin{aligned} S_1+kS_2 = {} & x^2+y^2+2g_1x+2f_1y+c_1 \\ & +k(x^2+y^2+2g_2x+2f_2y+c_2) \end{aligned} \qquad \ldots\ldots(3)$$

If (x_1, y_1) is a point of intersection of the circles (1) and (2), then we have

$$x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1=0$$

and $$x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2=0.$$

Substituting (x_1, y_1) in (3) we find

$$\begin{aligned} x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1+k(x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2) \\ =0+k(0)=0. \end{aligned}$$

Thus (x_1, y_1) is also on the graph of equation (3).

Now we rewrite (3) in the form,

$$(1+k)(x^2+y^2)+(2g_1+2kg_2)x+(2f_1+2kf_2)y+(c_1+kc_2)=0.$$

If $k \neq -1$, we may divide throughout by $1+k$ and rewrite the result in the form

$$x^2+y^2+\frac{2g_1+2kg_2}{1+k}x+\frac{2f_1+2kf_2}{1+k}y+\frac{c_1+kc_2}{1+k}=0$$

We now observe that this (and therefore (3) also) is the equation of a circle and, by the remarks above, this circle must contain the points of intersection of the given circles (1) and (2).

If $k = -1$, equation (3) reduces to the equation

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0. \qquad \cdots\cdots(4)$$

We have assumed that the circles (1) and (2) intersect, hence they are not concentric and we cannot have both $g_1 = g_2$ and $f_1 = f_2$ (why ?). Therefore (4) is an equation of the first degree and its graph is a line. Moreover, this line must contain the points of intersection of (1) and (2), so that if these circles intersect in two distinct points, the line (4) is the common chord.

Equation (3) is sometimes referred to as the equation of a family of circles, since we obtain different choices of the real number k.

Example. Find the equation of the circle which contains the points of intersection of the circles with equations

$$x^2+y^2+2x-4y-11=0$$

and $$x^2+y^2-2x-3=0$$

and which also contains the point $(3, -2)$.

Solution. Equation of the circle through the points of intersection of the given two circles can be written in the form

$$x^2+y^2+2x-4y-11+k(x^2+y^2-2x-3)=0.$$

If this circle is to pass through the point $(3, -2)$ we have

$$9+4+6+8-11+k(9+4-6-3)=0$$

$$i.e., k=-4.$$

Thus the required equation is

$$x^2+y^2+2x-4y-11-4(x^2+y^2-2x-3)=0$$

or $3x^2+3y^2-10x+4y-1=0.$

Exercise 8.3

1. Find the equation of the circle which contains the points of intersection of the given circles and also the given point.

   (*i*) $x^2+y^2+2x-1=0,\ x^2+y^2+x-y-4=0$; $(1,1)$

   (*ii*) $x^2+y^2+6x-4y-4=0,\quad x^2+y^2-4x-2y=0$; $(0, 6)$

   (*iii*) $3x^2+3y^2+4x-10y+2=0, 3x^2+3y^2-8x+2y=0$; $(2, 1)$.

2. Find the equation of the circle which contains the points of intersection of the circles $x^2+y^2+2x-1=0$, $x^2+y^2+x-y-4=0$ and has its centre on the line

$$x-2y-3=0.$$

3. If the circles with equations

$$x^2+y^2+2g_1\,x+2f_1\,y+c_1=0,$$
$$x^2+y^2+2g_2\,x+2f_2\,y+c_2=0$$

touch each other, show that

$$2\,(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0$$

is the equation of their common tangent.

4. Find the length of the common chord of the circles

$$x^2+y^2+6x-8y=1,\ x^2+y^2+4x-7y=4.$$

ANSWERS

1. (*i*) $5x^2+5y^2+7x-3y-14=0$

   (*ii*) $x^2+y^2+11x-5y-6=0$

   (*iii*) $9x^2+9y^2-18x-10y+1=0$

2. $x^2+y^2+10x+8y+23=0$

4. $\dfrac{18\sqrt{5}}{5}$

8.7. *The Length of Tangent Segment from a point.*

Let P (x_1, y_1) be the point and let PT be the tangent drawn from P to the circle

$$x^2+y^2+2gx+2fy+c=0.$$

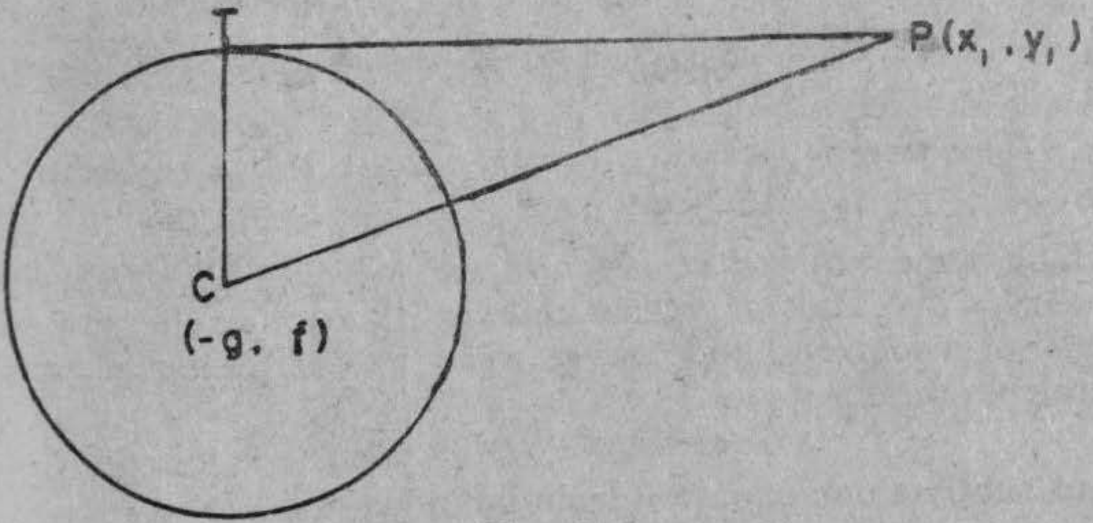


Fig. 8.3

From the right triangle PCT, we bave

$$PT^2=PC^2-CT^2 \text{.............. ....} \quad (I)$$

Now $PC^2=(x_1+g)^2+(y_1+f)^2$ and

$$CT=\text{radius of the circle}=\sqrt{g^2+f^2-c.}$$

Thus from (I) we get

$$PT^2=(x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$$
$$=x_1{}^2+y_1{}^2+2gx_1+2fy_1+c,$$

which is the square of the length of the tangent from

P (x_1 , y_1) to the circle $x^2+y^2+2gx+2fy+c=0.$

Definition. Let $S_1=0$ and $S_2=0$ be the equations of two circles. The locus of all points P, for which the length of the tangent from P to $S_1=0$ is the same as the length of the tangent from P to $S_2=0$, is called the *radical axis* of the circles $S_1=0$ and $S_2=0$.

8.7.1. *The Equation of the Radical Axis.*

Let the circles be $S_1=x^2+y^2+2g_1x+2f_1y+c_1=0$

and $S_2=x^2+y^2+2g_2x+2f_2y+c_2=0$

and suppose that P (x_1, y_1) be a point such that the length of the tangent from P to $S_1=0$ i the same as the length of tangent from P to $S_2=0$. Then by § 8.7, we have

$$x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1=x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2,$$

or $2(g_1-g_2)x_1+2(f_1-f_2)y_1+(c_1-c_2)=0$.

This proves that the point P (x_1, y_1) lies on the locus

$$2(g_1-g_2)x+2(f_1-f_2)y+(c_1-c_2)=0,$$

which is the required equation of the radical axis of $S_1=0$ and $S_2=0$ Thus in order to obtain equation of the radical axis of twogiven circles, we write the equation of each circle in the form

$$x^2+y^2+2gx+2fy+c=0$$

and subtract one equation from the other.

Note that if two circles intersect, their radical axis is the line through their two points of intersection *i.e* it is the common chord of the given circles. On the other hand, if two circles do not intersect, their radical axis does not intersect either of the circles. In case the circles touch each other, the radical axis is the common tangent to the circles.

Example 1. Find the length of the tangent from (8,5) to the circle of radius 3 with cenre (1,2).

Solution. The given circle is

$(x-1)^2+(y-2)^2=9$

or $x^2+y^2-2x-4y-4=0$.

Thus length of the tangent from (8, 5) to this circle

$=\sqrt{(8)^2+(5)^2-2(8)-4(5)-4}$

$=\sqrt{64+25-16-20-4}=\sqrt{49}=7$.

Example 2. Find the radical axis of the two circles

$$(x-2)^2+y^2=9$$

and $$(x-8)^2+y^2=25.$$

Solution.—We write the equations of the given circles in the form

$$x^2-4x+4+y^2-9=0$$

$$x^2-16x+64+y^2-25=0.$$

Subtracting $12x-60+16=0$

Hence the equation of the radical axis is

$$3x-11=0.$$

8.8. *Coaxal Circles.*

Definition—A system of circles is called coaxal if every two members of the system have the same radical axis.

8.8.1. *Theorem.* If

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad (1)$$

and $$S_2=x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad (2)$$

be two circles then

$$S_1+kS_2=0 \qquad (3)$$

represents for all values of k $(k \neq -1)$ a system of circles coaxal with the given circles.

Proof :—

For any value of k $(k \neq -1)$ (3) is

$$(1+k)x^2+(1+k)y^2+2(g_1+kg_2)x+2(f_1+kf_2)y+(c_1+kc_2)=0$$

or $$x^2+y^2+2\frac{(g_1+kg_2)}{1+k}x+2\frac{(f_1+kf_2)}{1+k}y+\frac{c_1+kc_2}{1+k}=0,$$

which is a circle.

Two members of the system, can be obtained by giving two different values k_1 and k_2 to k *i e.*

$$x^2+y^2+2\frac{(g_1+k_1g_2)}{1+k}x+2\frac{(f_1+k_1f_2)}{1+k_1}y+\frac{c_1+k_1c_2}{1+k_1}=0$$

and $$x^2+y^2+2\frac{(g_1+k_2g_2)}{1+k_2}x+2\frac{(f_1+k_2f_2)}{1+k_2}y+\frac{c_1+k_2c_2}{1+k_2}=0.$$

Their redical axis is obtanied by subtraction and on simplfication it is found to be.

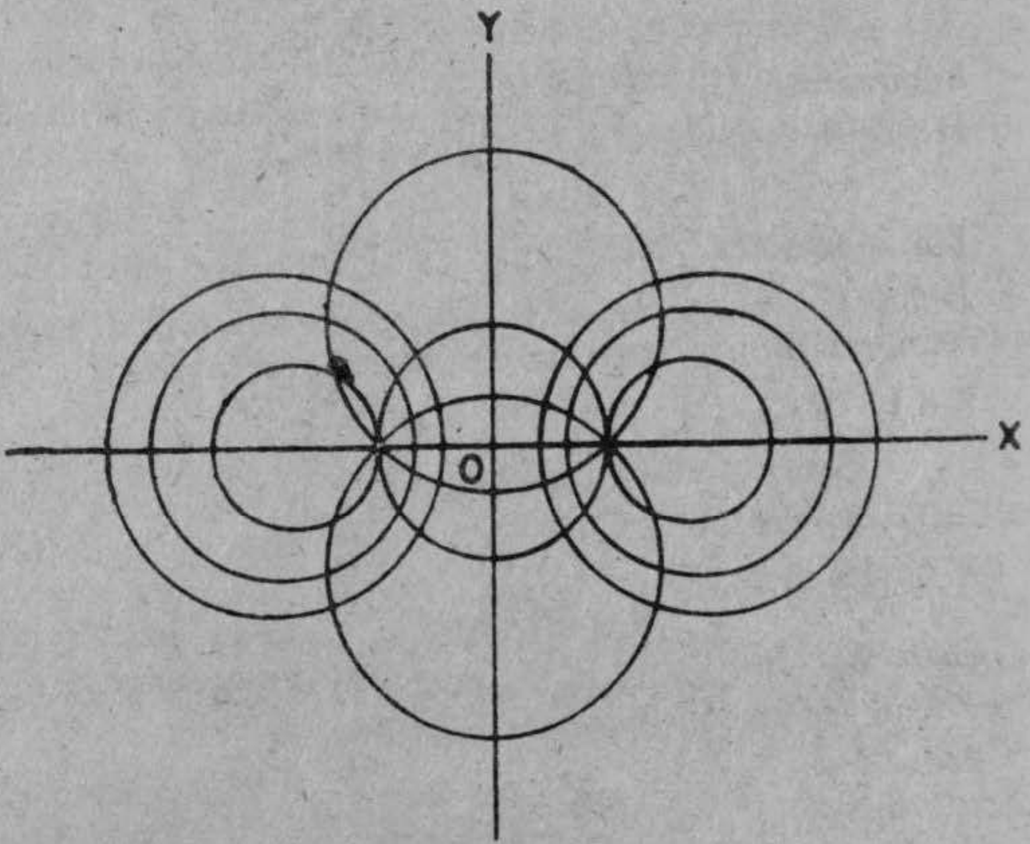


Fig. 8.4

$$2\,(g_1-g_2)\,x+2\,(f_1-f_2)\,y+c_1-c_2=0,$$

which is the same as the radical axis of (1) and (2).

Hence radical axis of any two members of the system is the same.

Note (1). If $k=-1$, then (3) becomes the radical axis of the system.

Note (2). If $S_1=0$ and $S_2=0$ intersect them, (refer § 8 6)

$$S_1+kS_2=0$$

represents a system of circles containing the points of intersection of $S_1=0$ and $S_2=0$.

Hence all members of the coaxal system contain two common points and the system is called intersecting type.

The radical axis of the intersecting type of the coaxal system of circles also passes through their common points.

If circles (1) and (2) do not intersect then the system of circles (3) is also of the non-intersecting type and has members lying on both sides of the radical axis. In this case the system consists of two families of circles. Each family has also circle of zero radius as its member. These point circles are called limiting points of the non-intersecting system of coaxal circles.

Example 1 Find the circle coaxal with

$$x^2+y^2-3x+y-1=0 \text{ and}$$
$$x^2+y^2+2x-y+5=0$$

and passing through (1, 1).

Solution. Any circle coaxal with the given circle is $(x^2+y^2-3x+y-1)+k(x^2+y^2+2x-y+5)=0$.
This passes through (1, 1) if

$$(1+1-3+1-1)+k(1+1+2-1+5)=0$$

or $$-1+8k=0$$

$$\therefore \quad k=\tfrac{1}{8}.$$

Hence the required circle is

$$(x^2+y^2-3x+y-1)+\tfrac{1}{8}(x^2+y^2+2x-y+5)=0$$

or $$9x^2+9y^2-22x+7y-3=0.$$

Example 2. Discuss the system of circles coaxal with

$$x^2+y^2-10x+16=0,\ x^2+y^2+10x+16=0.$$

Solution. Subtracting the equations of given circles we have

$$-20x=0 \quad \text{or } x=0$$

as the equation of the radical axis.

Putting $x=0$ in the equation of the first circle we get

$$y^2+16=0,$$

which is impossible. Hence the circles do not intersect. The coaxal system is

$$x^2+y^2-10x+16+k(x^2+y^2+10x+16)=0$$

$$\text{or} \quad (1+k)(x^2+y^2)-10(1-k)x+16(1+k)=0$$

$$\text{or} \quad x^2+y^2-10\,\frac{1-k}{1+k}x+16=0.$$

The centre of this circle is

$$\left(\frac{5\,(1-k)}{1+k},0\right) \text{ and radius is } \left[\frac{25\,(1-k)^2}{(1+k)^2}-16\right]^{\frac{1}{2}}$$

The circle reduces to a point if $\frac{5(1-k)}{1+k}=\pm 4$ or

$$5\,(1-k)=\pm 4\,(1+k)$$

$$\text{or} \quad k=\tfrac{1}{9},\ 9.$$

Therefore the limiting points of the system are $(4, 0)$, $(-4, 0)$.

Exercise 8.4.

1. Find the length of the tangent from $(4, 1)$ to the circle $x^2+y^2+6x-4y-12=0$.

2. Find the radical axis of the two circles

$$x^2+(y-2)^2=4,\ x^2+(y-3)^2=11.$$

3. Find the point from which the tangents to each of the following three circles are of equal length

$$x^2+y^2+8x+6y=0.\ x^2+y^2+9x+3y=1,$$
$$x^2+y^2+7x+4y=-9.$$

4. Find the circle coaxal with each of the circles

$$x^2+y^2-2y-1=0,\ 3x^2+3y^2+4y-2=0 \text{ and}$$

containing the point (4, 2).

(5) (*a*) Find the radical axis of the circles

$$x^2+y^2-6x-4y+5=0,\ x^2+y^2+8x-2y=1$$

and prove that it is tangent to both of them. Draw the figure.

(*b*) Find the circle coaxal with these two circles and contatining the origin.

6. Find the family of all circles which have the line $x+2y=20$ as radical axis with the circle

$$x^2+y^2-4x+2y=75.$$

7. Prove analytically that the radical axis of two circles is perpendicular to their line of centres.

ANSWERS

1. 5

2. $y=-1$

3. $(\frac{29}{5}, \frac{8}{5})$

4. $x^2+y^2-\frac{64}{7}y-\frac{12}{7}=0$

5. (*a*) $7x+y=3$

(*b*) $3x^2+3y^2+17x-7y=0$

6. $x^2+y^2-(4-k)x+2(1+k)y-(75+20k)=0$

Definition. A *conic* is the locus of all points whose distance from a fixed point bears a constant ratio to their distance from a fixed line.

The fixed point is called the *focus*, the fixed line is said to be the *directrix* and the constant ratio, which is denoted by e, is known as the eccentricity.

If $e = 1$, the conic is a parabola.

If $e < 1$, the conic is an ellipse.

If $e > 1$, the conic is a hyperbola.

Thus (by definition of a conic) a parabola is the locus of all points whose distance from the focus is equal to their distance from the directrix.

8 9. *Standard Equation of a parabola.*

To derive the equation of a parabola in the simplest form we assume that the focus S is a point on the x — axis having coordinates $(a, 0)$. We also suppose that the directrix AB is a line parallel to y — axis having equation $x + a = 0$.

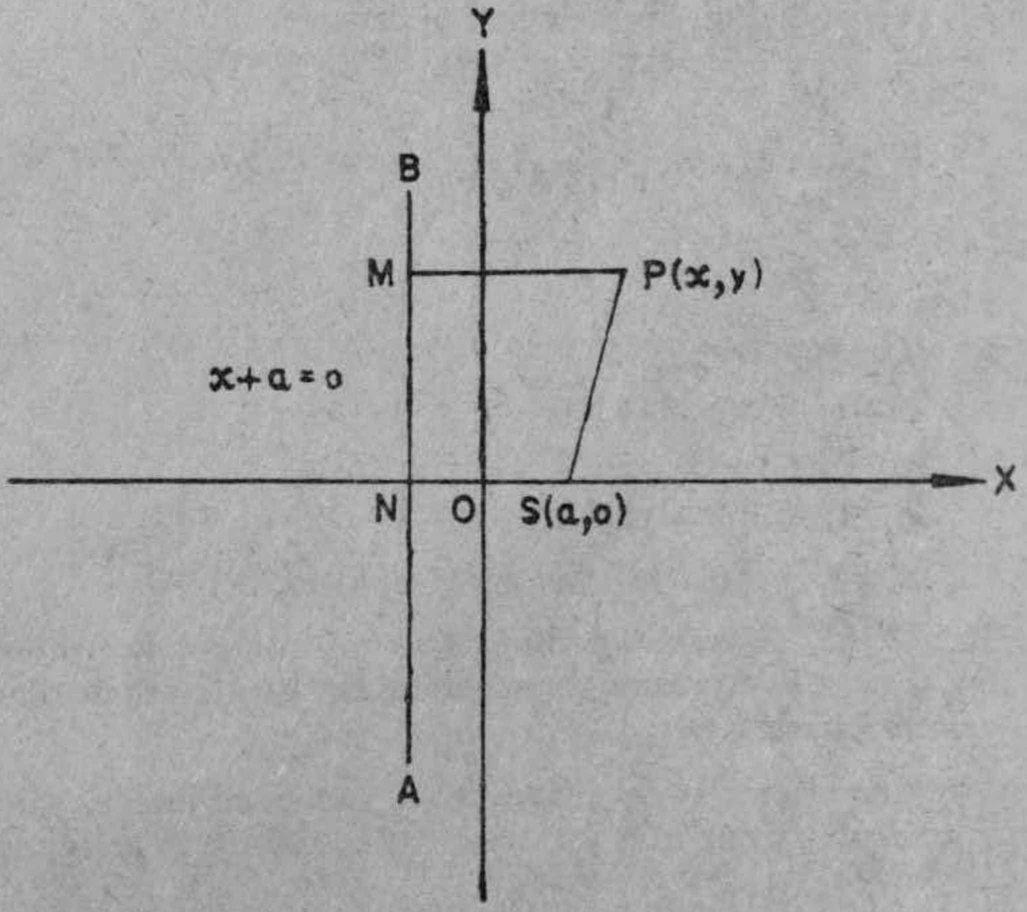


Fig. 8.5

Let P (x, y) be a point on the parabola. Draw PM perpendicular from P to A B. By definiton

$$P S = P M,$$

or $\sqrt{(x-a)^2+y^2} = \frac{x+a}{\sqrt{1^2}}$ Squaring both sides we get

$$(x-a)^2+y^2=(x+a)^2,$$

or $\quad y^2+x^2-2ax+a^2=x^2+2ax+a^2,$

or $\quad y^2=4ax$

as the standard equation of a parabola.

8.9.1. *Definitions*

(*i*) The mid point of the prependicular from the focus to the directrix is called the *vertex* of the parabola. In figure 8.5. O is the vertex of the parabola $y^2 = 4ax$.

(*ii*) The line through the focus perpendicular to the directrix is called the *axis* of the parabola.

Clearly the axis meets the parabola at the vertex.

In figure 8.5, x — axis is the axis of the parabola $y^2 = 4ax$.

From the equation $y^2 = 4ax$ we infer the following about this parabola

(*i*) (0, 0) lies on the parabola.

(*ii*) It is symmetric about its axis.

(*iii*) Since $y^2 \geqslant 0$ for all real numbers y, we must have $4ax \geqslant 0$. This in turn means tnat $x \geqslant 0$ if $a > 0$, and $x \leqslant 0$ if $a < 0$. Thus if $a > 0$, the parabola is to the right of the y—axis, whereas if $a<0$, the parabola is to the left of the y—axis.

(*iv*) For large values of $|x|$, y^2 is large and the parabola is far from the x—axis. We may thus conveniently say that the parabola opens to the right if $a>0$, and opens to the left if $a<0$.

(v) If $x \to \infty$, $y \to \infty$.

Thus we may conclude that (if $a > 0$) the shape of the parabola $y^2 = 4ax$ is as shown below.

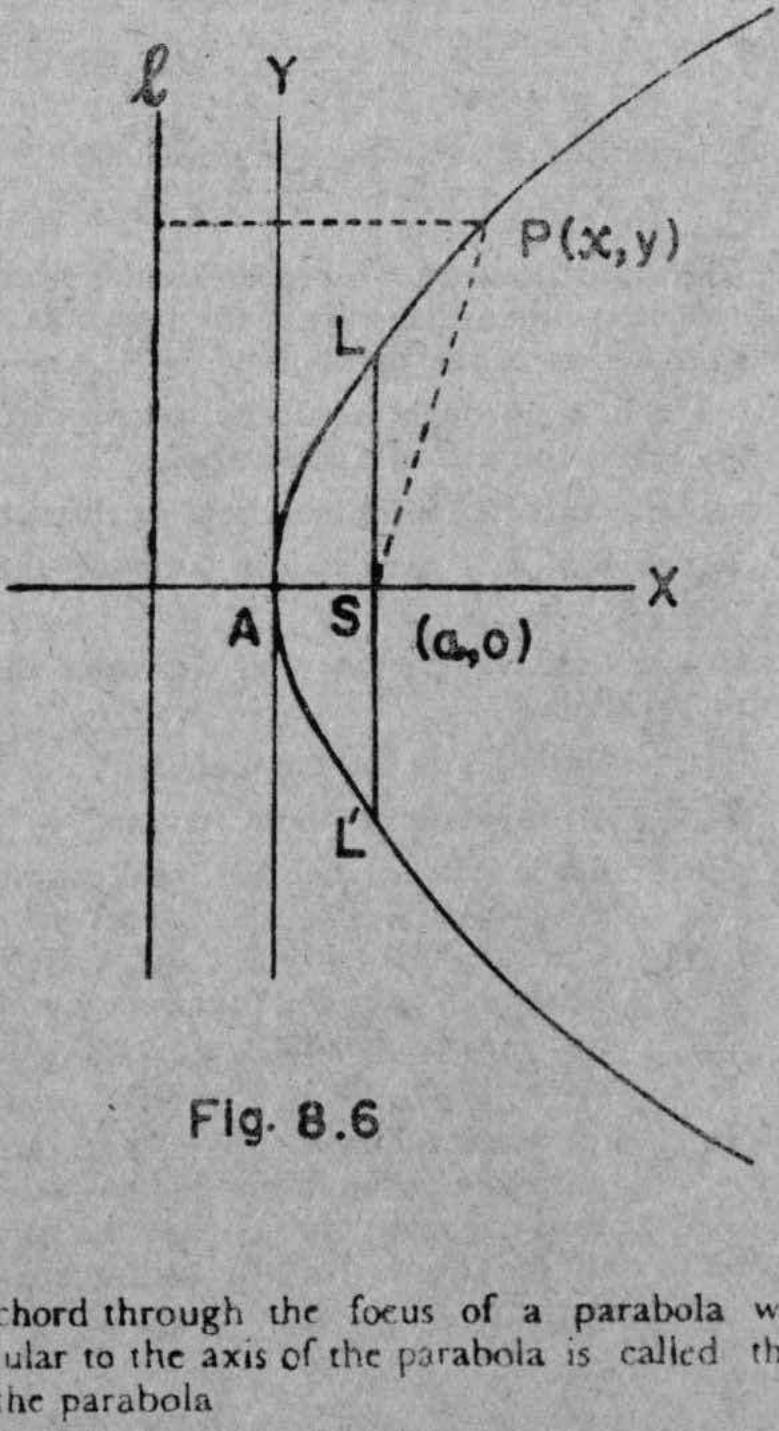


Fig. 8.6

The chord through the focus of a parabola which is perpendicular to the axis of the parabola is called the *latus rectum* of the parabola

In Figure 8.7. the latus rectum is CD. Let E and A be the respective projections of Dand F on the directix.

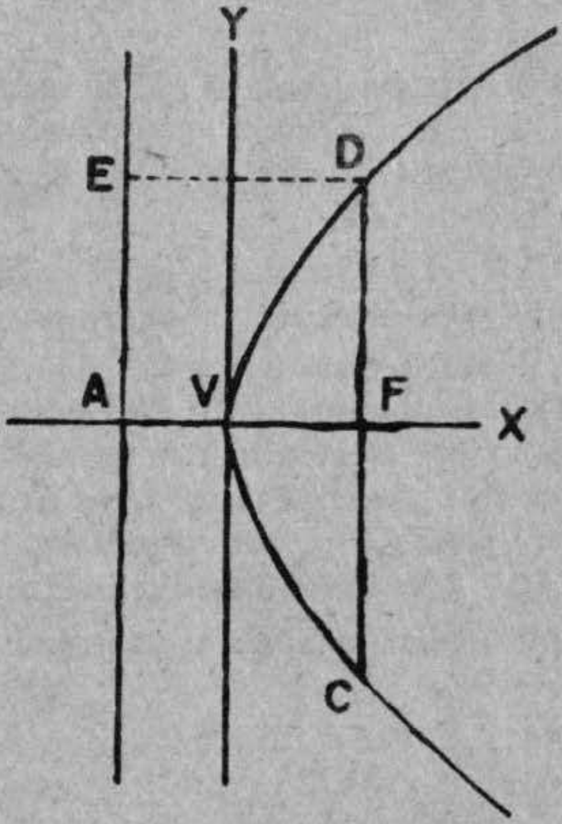


Fig. 8.7

Then $|ED| = |DF|$ by definition of the parabola. However, $|ED| = |AF| = |2a|$. Thus the total length $|CD|$ of the latus rectum is $|4a|$ *i.e.* the length of the latus rectum of the parabola (1) is just the absolute value of the coefficient of x in its equation.

An approximate idea of the shape of a parabola may be obtained from a knowledge of the vertex and the extremities of the latus rectum. To draw a more accurate graph, one should also plot additional points.

In the derivation of the equation of a parabola, if we had chosen the axis of the parabola to be the y-axis and the vertex to be the origin, the effect would simply be to interchange the roles of x and y. Thus the equation of a parabola with the origin as its vertex and with the y-axis as its axis is

$$x^2 = 4ay \qquad \dots \dots \dots \dots \qquad (2)$$

The focus of this parabola is $(0, a)$ and the equation of its directrik is $y = -a$. Clearly the parabola opens upward if $a > 0$ and downward if $a < 0$. The length of the latus rectum is still $|4a|$.

Example 1. For the parabola $x^2 = 4y$, find the focus, the directrix and latus rectum. Sketch its graph.

Solution. —Since $4a = 4$, $a = 1$. The coordinates of the focus are $(0, 1)$. Equation of the directrix is $y + 1 = 0$, the latus rectum is 4. The vertex of this parabola is at the origin, its axis lies on the y-axis and it opens upward. To obtain two additional points on the parabola, put $y = 1$, in the equation $x^2 = 4y$. We thus obtain $x = \pm 2$.

Thus graph of the parabola is as given in Figure 8.8.

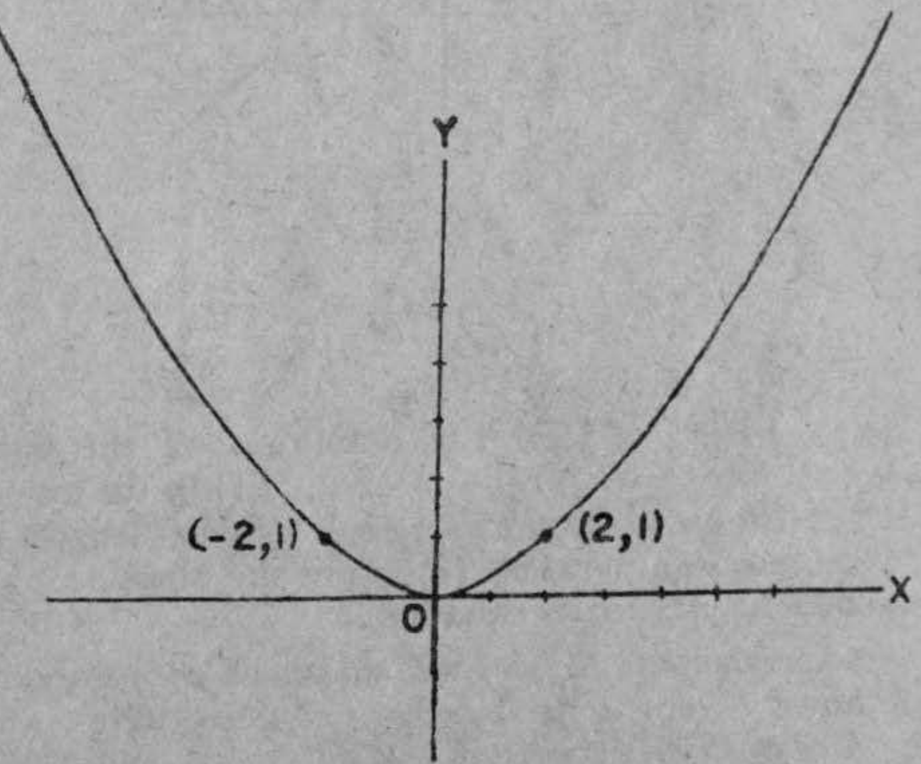


Fig. 8.8

Example 2. Draw the parabola $2y^2 = -x$.

Solution. The given equation can be written in the form $y^2 = -\frac{1}{2}x$,

which is of the form of equation (1). The vertex of this parabola is also at the origin, while its axis lies along the x – axis and it opens to the left. Further if

$$x = -2.$$

$$\text{then} \quad y = \pm 1.$$

Thus the graph of the parabola is as given in Figure 8.9.

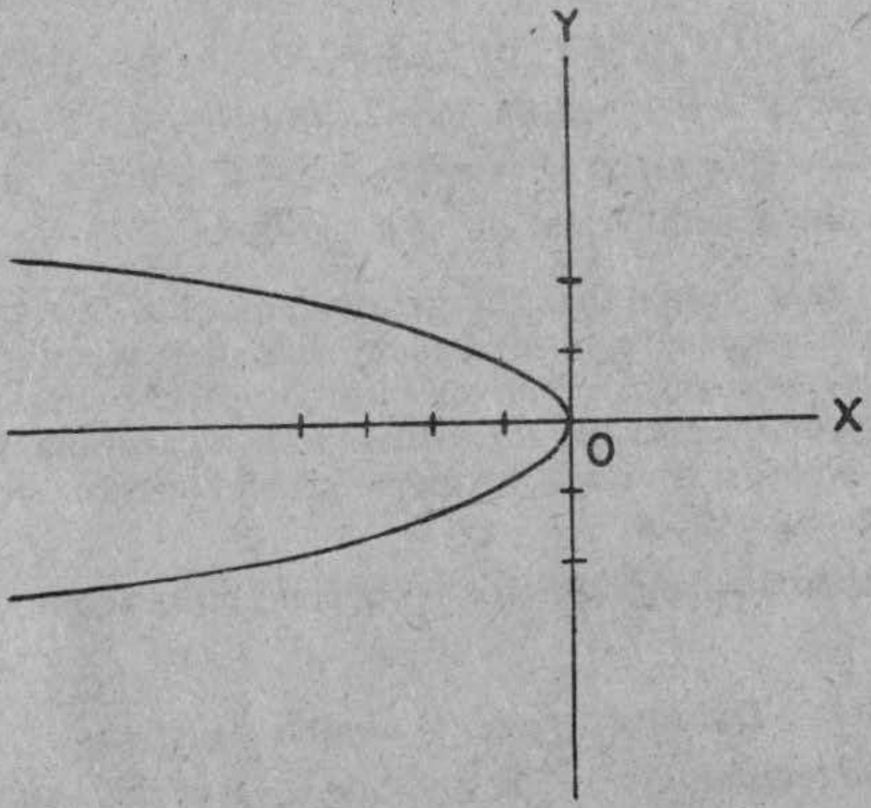


Fig. 8.9

Example 3. Find the equation of the parabola whose tex is at the origin and whose focus is at the point (0, −2).

Solution. Here $a=-2$, $4a=-8$ and the equation f the form $x^2=4ay$.

Therefore the required equation is $x^2=-8y$.

8.10. *General Equation of a Parabola*

Let S (x_1, y_1) be the focus and $lx+my+n=0$ be the directrix. If P (x, y) be a point on the parabola, then (by definition)

$|PS|$ = length of perpendicular from P on $lx+my+n=0$, so that $\sqrt{(x-x_1)^2+(y-y_1)^2}=\frac{lx+my+n}{\sqrt{l^2+m^2}}$,

Squaring both sides we get

$$(x^2+y^2-2x_1x-2y_1y+x_1^2+y_1^2)(l^2+m^2)$$
$$=l^2x^2+m^2y^2+n^2\mp 2\,lm\,xy+2\,ln\,x+2mn\,y$$

or $$m^2x^2+l^2y^2-2\,lm\,xy-2x(ln+x_1+m^2)$$
$$-2y\{mn+y_1(l^2+m^2)\}+\{n^2-(l^2+m^2)(x_1^2+y_1^2)\}=0;$$

which is the general equation of a parabola.

It is to be noted that the general equation of a parabola is of the second degree and is such that the second degree terms, viz. $m^2x^2+l^2y^2-2\,lm\,xy$ form a perfect square. This characteristic property of the equation of a parabola is used to prove whether or not a given second degree equation represents a parabola.

Further discussion of this is beyond the scope of this book.

8.10.1. *Deductions from the General Equation*

In the equation $y^2=\ ax$ the line $y=0$ is the axis of the parabola and the origin (0, 0) is the vertex of parabola. The equation

$$(y-k)^2=4a(x-h) \qquad \text{.........(1)}$$

represents a parabola whose vertex is at the point (h, k) and whose axis is $y-k=0$.

Similarly, the equation

$$(x-h)^2=4a\,(y-k) \qquad \cdots\cdots\cdots(2)$$

represents a parabola whose vertex is at the point (h, k) and whose axis is $x-h=0$.

In general the equation

$$ax^2+by^2+2gx+2fy+c=0 \qquad \cdots\cdots\cdots(3)$$

represents a parabola if either

(*i*) $a=0$ and $b\neq 0$ (*ii*) $a\neq 0$ and $b=0$.

For if $a=0$ and $b\neq 0$, we can write (3) in the form

$$b\left(y^2+\frac{2f}{b}y+\frac{f^2}{b^2}\right)=-2gx-c+\frac{f^2}{b}$$

$$\text{or}\left[y-\left(-\frac{f}{b}\right)\right]^2=-\frac{2g}{b}\left[x-\left(\frac{bc-f^2}{2gb}\right)\right]$$

which is of the form (1).

Similarly in case (*ii*) we have a parabola.

Example 1. Find the vertex and axis of the parabola

$$x^2-6x-2y+5=0.$$

Solution: $x^2-6x-2y+5=0$

$$\Rightarrow (x-3)^2=2(y+2).$$

This represents a parabola whose vertex is at the point $(3,-2)$ and whose axis is the line $x-3=0$.

Example 2. Find the equation of the parabola whose focus is $(1, 2)$ and directrix is the line $x+y=t$.

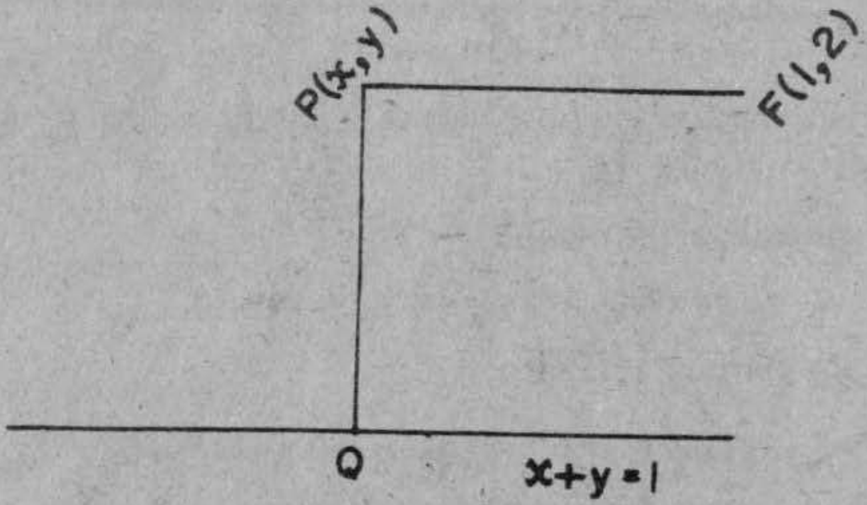


Fig 8.10

Solution. Let P (x, y) be any point on the parabola. Then distance of P (x, y) from $(1, 2)$

$$=\sqrt{(x-1)^2+(y-2)^2}.$$

Now perpendicular distance of P (x, y) from the directrix $x+y=1$ is

$$\frac{|x+y-1|}{\sqrt{2}}.$$

By definition the equation of the parabola is

$$\frac{|x+y-1|}{\sqrt{2}}=\sqrt{(x-1)^2+(y-2)^2}$$

or $\quad (x+y-1)^2=2\,[(x-1)^2+(y-2)^2].$

Exercise 8.5

1. Draw the following parabolas

(*i*) $y^2=6x$ (*ii*) $x^2=8y$

(*iii*) $y^2=5x$ (*iv*) $3x^2=-y$.

2. Find the equation of the parabolas which satisfy the following conditions :

(*i*) vertex $(0, 0)$; focus $(2, 0)$

(*ii*) vertex $(0, 0)$; focus $(0,-4)$

(*iii*) focus $(-1, 0)$; directrix $x-1=0$

(*iv*) vertex at origin ; focus $(0, \frac{1}{2})$

(*v*) vertex $(0, 0)$; contains the point $(8, 1)$.

3. Find the equation of the circle whose diameter is the latus rectum of the parabola

$$x^2=-16y.$$

4. Find the point of intersection of the parabola

$$y^2=4x \text{ and the line } x-2y+3=0.$$

5. Find the points of intersection of the two parabolas

$$y^2=3x \text{ and } x^2=24y.$$

6. Find the equation of the parabola whose vertex is the point $(3, 2)$ and focus $(3, 4)$.

7 Find the equation of the parabola whose focus is $(-5, 3)$ and directrix $y=7$.

8. The distance between a point on a parabola and the focus of the parabola is called a focal radius of the parabola. Prove that the focal radius of the point (a, b) on the parabola $x^2=4cy$ equals $|\ b+a\ |$.

9. Draw the following parabolas

(*i*) $x^2+4x-y+5=0$ (*ii*) $y^2-x-2y-1=0$.

ANSWERS

2. (*i*) $y^2=8x$

(*ii*) $x^2=-16y$

(*iii*) $y^2=-4x$

(*iv*) $y^2=2x$

(*v*) $x^2=64y$; $y^2=\frac{1}{8}x$.

3. $x^2+y^2+8y-48=0$

4. (1, 2), (9. 6)

5. (0, 0), (12, 6)

6. $x^2-6x-8y+25=0$

7. $x^2+10x+8y-15=0$

8.11. *Equation of the Ellipse*

Let S be the focus and LM the directrix. Draw SZ perpendicular to LM. Produce ZS and divide it internally and externally at A and A′ respectively in the same ratio $e:1$ (e is the eccentricity). Bisect AA′ at O and let AA′ $=2a$,

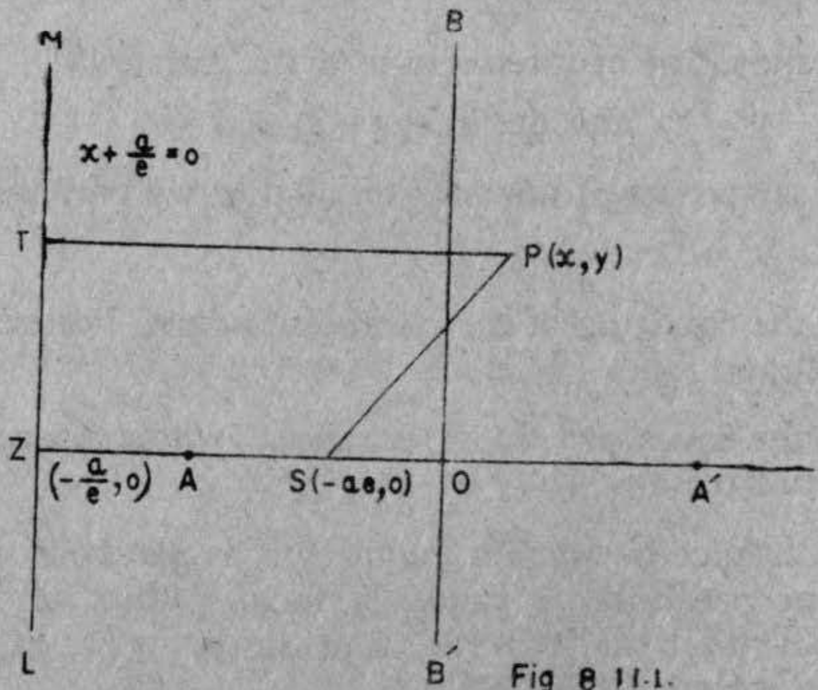


Fig 8 11·1.

so that AO $=$ OA′ $=a$. Through O draw BB′ perpendicular to AA′.

Now $\frac{AS}{ZA}=e$ and so $AS=e.\,ZA$ ......(*i*)

Also $\frac{SA'}{ZA'}=e$ and so $SA'=e.\ ZA'$ ......(ii)

Adding (i) and (ii). we get

$$AS+SA'=e\,(ZA+ZA'),$$

$$\text{or } AA'=e\,\{ZA+(ZA+AA')\}$$

$$=e\,(2ZA+2\,AO),\ (\because\ AA'=2\,AO)$$

$$=2\,e\,(ZA+AO)=2\,e.\ ZO.$$

Thus $2a=2e.\ ZO$ and hence $ZO=\frac{a}{e}$.

Again subtracting (i) from (ii) we get

$$SA'-AS=e\,(ZA'-ZA),$$

$$\text{or }\ (SO+OA')-(AO-SO)=e\{(ZA+AA')-ZA\}$$

$$=e.\ AA'$$

$$=2\,e.\ AO,$$

$$\text{or }\ 2\,.\,SO=2\,e.\ AO,\ (\because\ \ OA'=AO=a)$$

$$=2\,e.\,a.$$

Thus $SO=ae$.

Taking O as the origin and AA′, BB′ as the axes of coordiates we see that the coordinates of S and Z are $(-ae, 0)$ and $\left(\frac{-a}{e}, 0\right)$ respectively.

Since the directrix LM is a line through $Z\left(-\frac{a}{e}, 0\right)$ parallel to y-axis, its equation is given by

$$x=-\frac{a}{e}\text{ or }\ x+\frac{a}{e}=0.$$

Let $P\,(x, y)$ be any point on the ellipse. Then by definition

PS = (e)(length of perpendicular from P to LM);

$$\text{or }\ \sqrt{(x+ae)^2+y^2}=e\left(x+\frac{a}{e}\right)$$

$$\text{or }\ (x+ae)^2+y^2=e^2\left(a+\frac{a}{e}\right)^2,$$

or $x^2+2ae\,x+a^2e^2+y^2=e^2x^2+2a\,ex+a^2$,

or $x^2(1-e^2)+y^2=a^2(1-e^2)$,

or $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$ $\Big(\because\ a\neq 0,\ 1-e^2\neq 0,$

we can divide by $a^2(1-e^2)\Big)$.

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $b^2=a^2(1-e^2)$.

This is the required standard equation of an ellipse.

Alternate Method. Let Fand F′ be two fixe points. Let $2c$ denote the distance between them and le

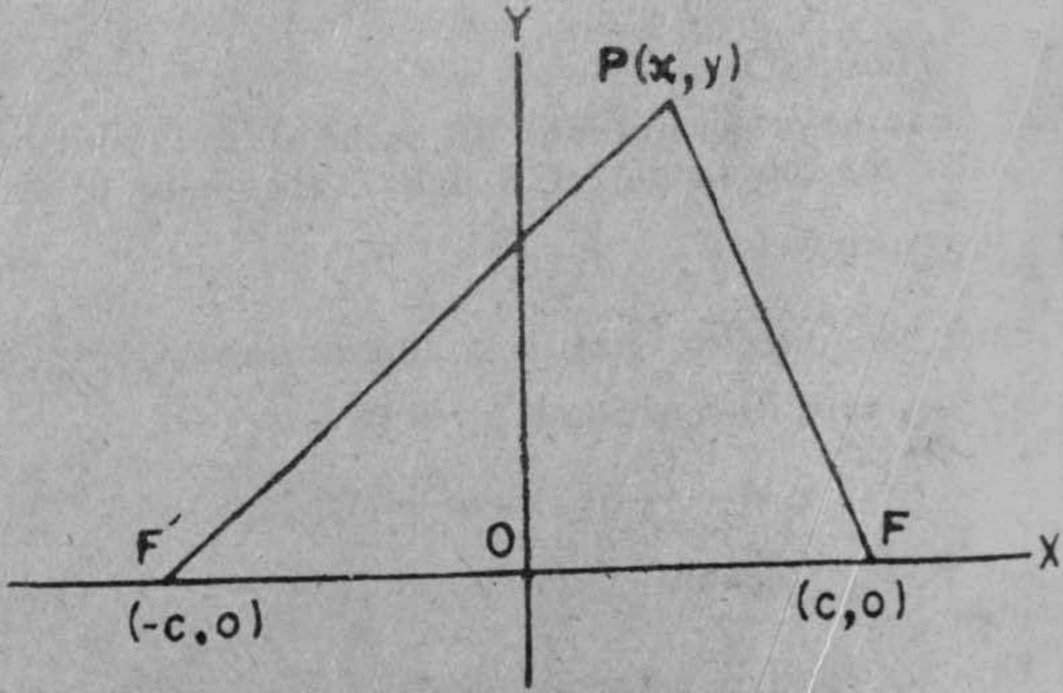


Fig. 8 11.2

'a' be a real number such that $a>c$. Then the locus of all points P such that

$$| FP | + | FP | = 2a$$

is called an *ellipse*.

Each of the two fixed points, F and F′ is called a *focus* of the ellipse The mid point O of the segment FF is called the *centre* of the ellipse. Since $| FF' | = 2c$, evidently $| OF | = | OF' | = c$. To derive the equation of an ellipse, we take the line joining the two foci F′ and F as x-axis and the line through O perpendicular to FF′ as y-axis. Then the foci F′ and F have coordinates $(-c, 0)$ and $(c, 0)$ respectively. Now the point P (x, y) lies on the ellipse if

$$| FP | + | F'P | = 2a$$

that is, if

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a.$$

Now $$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}.$$

Squaring both side we have

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

or $a\sqrt{(x-c)^2+y^2}=a^2-cx.$

Squaring again and simplifying we have

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2).$$

Dividing both sides of this equation by $a^2(a^2-c^2)$, we have

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$ as the equation of an ellipse.

Since $a>c$, $a^2-c^2>0$

and we may write

$$a^2-c^2=b^2. \tag{1}$$

Hence the above equation can be written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{2}$$

This is the standard equation of an ellipse.

Our calculation shows that for any point P (x, y) on the ellipse, the coordinates x and y of P satisfy (2) and conversely.

From equation (2) we can establish the following properties of the ellipse.

(*i*) The graph of (2) is symmetric with respect to the x-axis and the y-axis.

(*ii*) It is also symmetric with respect to the origin, *i.e.*, it is symmetric with respect to its centre.

(*iii*) The intersection of the ellipse (2) with the x-axis are the points A $(a, 0)$ and A′ $(-a, 0)$ and with the y-axis are B $(0, b)$ and B′ $(0, -b)$.

(*iv*) Since, by (1)

$$a^2 = b^2 + c^2.$$

it follows that $a > b$.

The points A and A′ are called vertices of the ellipse, the segment AA′ is called the *major axis* and the segment **BB′** the *minor axis* of the ellipse.

OA (OA′) is called the *semi-major axis* and **OB (OB′)** the *semi-minor axis*.

The length of the major axis is $2a$, and the length of the minor axis is $2b$.

If we solve (2) for y, we obtain

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2} \qquad \text{.........(3).}$$

From this it follows that if $|x| > a$, there is no corresponding real value of y, and hence the ellipse lies between the lines $x = a$ and $x = -a$. A similar calculation also shows that the ellipse lies between the lines $y = b$ and $y = -b$. Thus the ellipse lies in the rectangle determined by these four lines.

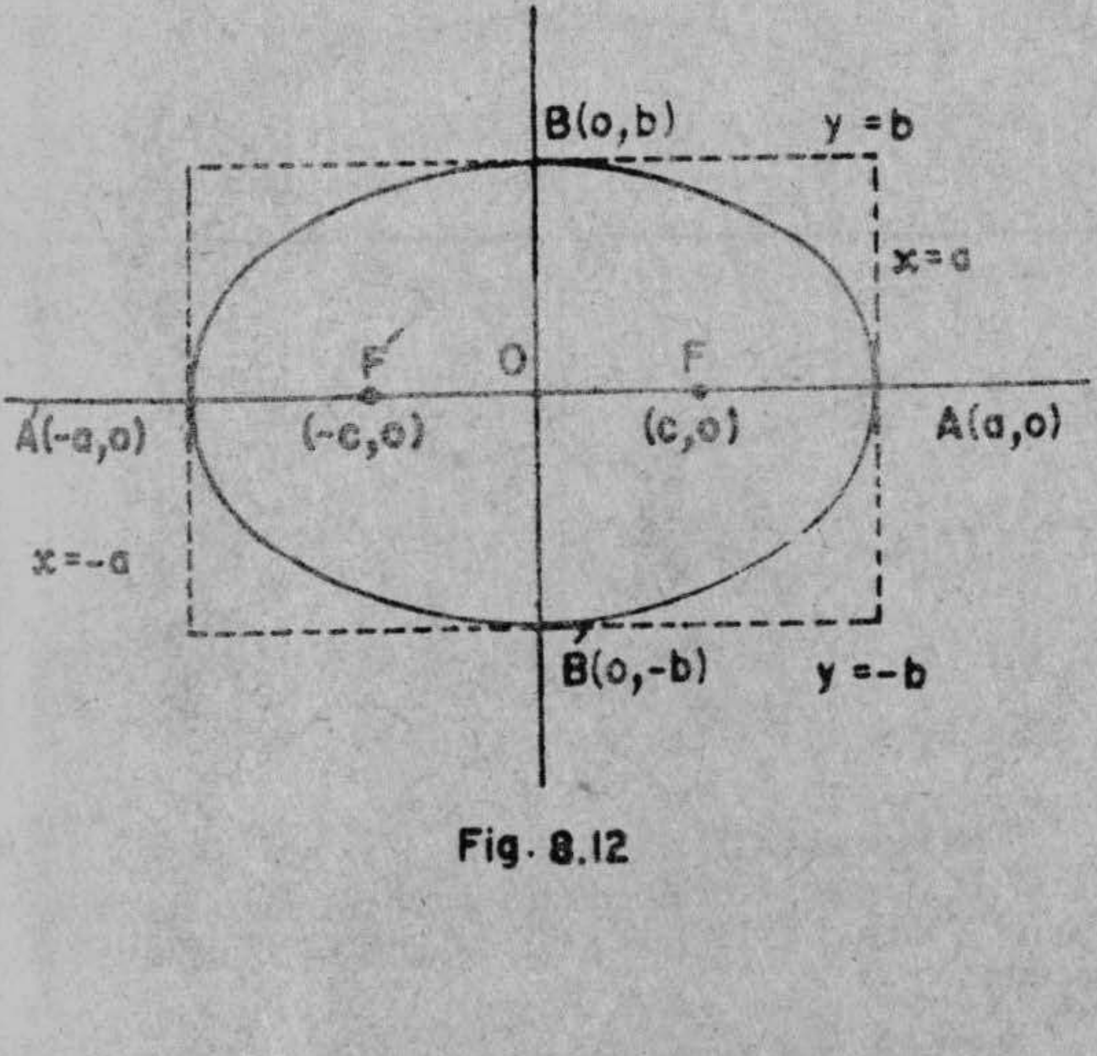


Fig. 8.12

We can define vertices, major and minor axes of an ellipse without reference to coordinate axes. The *vertices* are the points of intersection of the ellipse and the line joining the foci. The segment joining the vertices is the major axis and the minor axis is the segment cut by the ellipse from the line through the centre and perpendicular to the major axis.

The line through a focus F of an ellipse perpendicular to the major axis cuts the ellipse in two points, say C and D

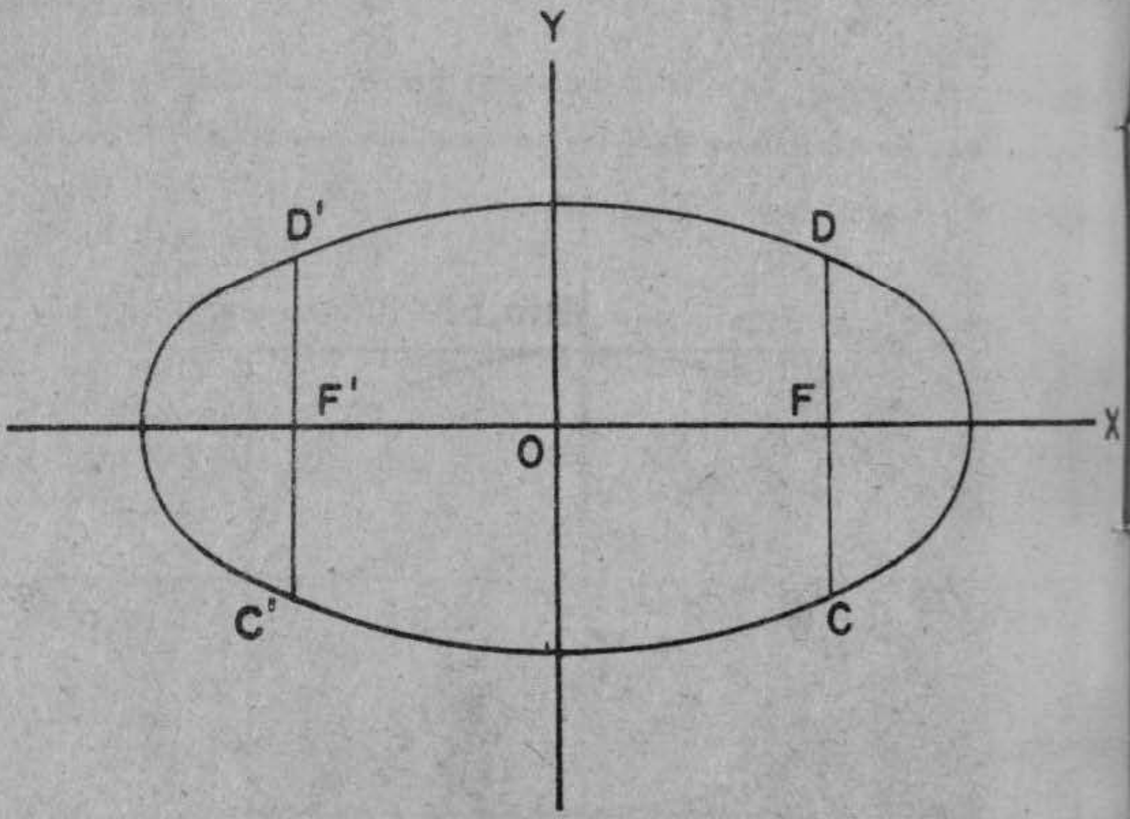


Fig. 8.13

The segment CD is called a *latus rectum* of the ellipse. If C′ D′ is the corresponding segment through the other focus F′, then | C′D′ | = | CD | , and C′ D′ is also a latus rectum.

To find the *y*-coordinates of the points C and D, we substitute $x = c$ in (3) and find that

$$y = \pm \frac{b}{a} \sqrt{a^2 - c^2}.$$

But by (1) $\sqrt{a^2 - c^2} = b$.

$$\text{Hence } y = \pm \frac{b^2}{a}.$$

Thus D is the point $\left(c, \frac{b^2}{a}\right)$ and C is the point $\left(c, -\frac{b^2}{a}\right)$.

Similarly D′ and C′ have coordinates $\left(-c, \frac{b^2}{a}\right)$ and $\left(-c, \frac{-b^2}{a}\right)$ respectively. It is clear that the length | CD | or | C′D′ | of a latus rectum is $\frac{2b^2}{a}$.

If we choose the line through the foci of the ellipse to be y-axis, and the centre to be the origin, the effect would simply be to interchange the roles of x and y. Thus equation of the ellipse with its centre at the origin and its foci on the y-axis is

$$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1 \qquad \text{.........(4).}$$

As before, the major axis (along the v-axis now) has length $2a$, the minor axis has length $2b$, the distance between the foci is $2c$, and length of each latus rectum is $\frac{2b^2}{a}$. The foci are now $(0, c)$ and $(0, -c)$ and vertices are $(0, a)$ and $(0, -a)$.

To draw the approximate shape of an ellipse from its equation, it is sufficient to locate the vertices, the ends of minor axis, and the ends of the latera recta.

In equation (2), putting $b=a$, we get

$$x^2+y^2=a^2$$

as the equation of the circle, whose centre is the origin and whose radius is a.

Thus circle is a special case of an ellipse whose major axis is equal to its minor axis.

Example 1. Find the semi-axes and the coordinates of the foci of the ellipse $9x^2+25y^2-225=0$. Also draw its graph.

Solution. The given equation can be written in the form

$$\frac{x^2}{25}+\frac{y^2}{9}=1.$$

Here $a=5$ and $B=3$.

Since $c^2=a^2-b^2=25-9=16,\ c=4$.

Thus the coordinates of the foci are $(4,\ 0)$ and $(-4,\ 0)$, the vertices are at $(-5,\ 0)$ and (5.0) and the ends of the minor axis are at $(0,-3)$ and $(0,\ 3)$. **The** curve is drawn in figure 8.14.

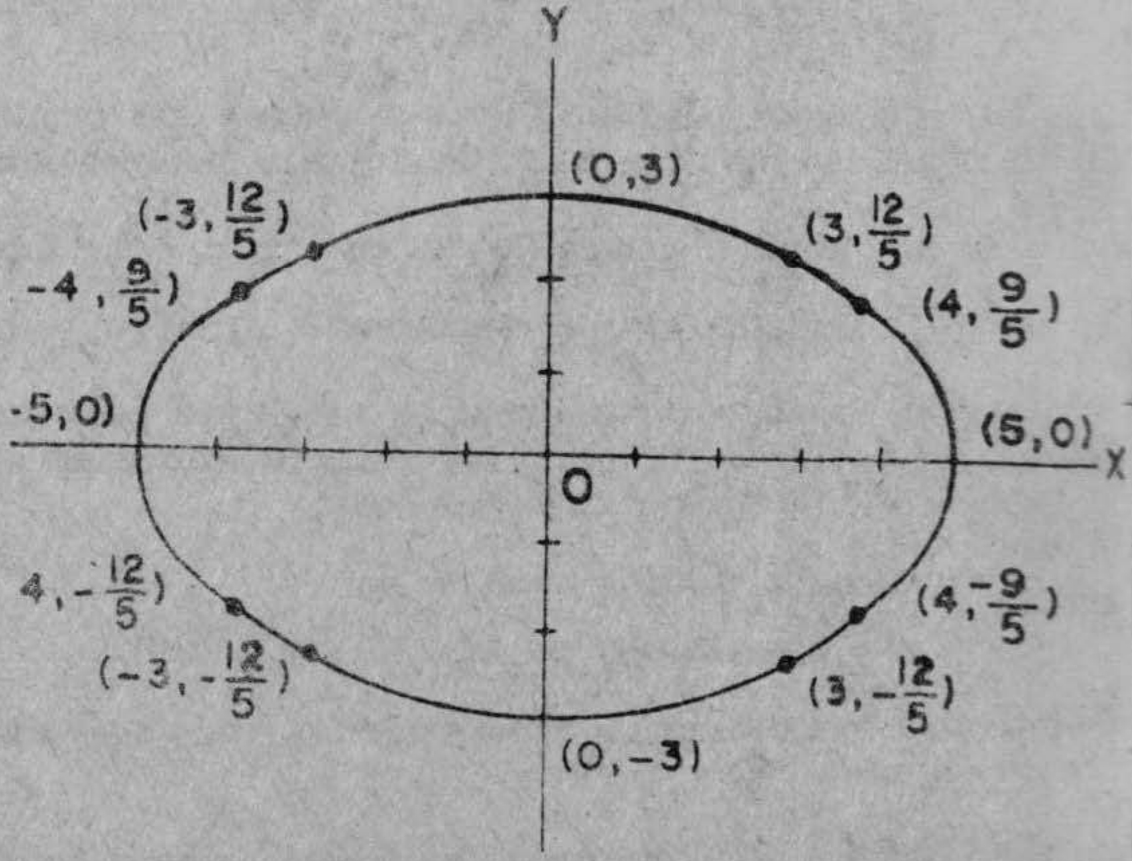

Y
(0,3)
(-3, $\frac{12}{5}$)
(3, $\frac{12}{5}$)
-4 , $\frac{9}{5}$)
(4, $\frac{9}{5}$)
-5,0)
(5,0)
X
O
4, -$\frac{12}{5}$)
(4, $\frac{-9}{5}$)
(-3, -$\frac{12}{5}$)
(3, -$\frac{12}{5}$)
(0,-3)

Fig. 8.14

Example 2. Draw the graph of the equation

$$16x^2+9y^2=36.$$

Solution. Dividing both sides of the equation by 36 we obtain

$$\frac{4x^2}{9}+\frac{y^2}{4}=1$$

and this can be written in the form

$$\frac{x^2}{\frac{9}{4}}+\frac{y^2}{4}=1$$

Since $4>\frac{9}{4}$ this equation has the form (4) with $a^2=4$ and $b^2=\frac{9}{4}$. By (1)

$$c^2=a^2-b^2=\frac{7}{4},$$

hence we have $a=2$, $b=\frac{3}{2}$ $c=\frac{\sqrt{7}}{2}$. The major axis of this ellipse is along the y-axis, and hence the vertices are $(0,2)$ and $(0,-2)$. The ends of the minor axis are $(\frac{3}{2},0)$ and $(-\frac{3}{2},0)$ and the foci are $\left(0,\frac{\sqrt{7}}{2}\right)$ and $\left(0,-\frac{\sqrt{7}}{2}\right)$. Since the total length of each latus rectum is $\frac{2b^2}{a}$, we find that the ends of the latera recta are $\left(\pm\frac{9}{8},\pm\frac{\sqrt{7}}{2}\right)$. The graph is in figure 8.15.

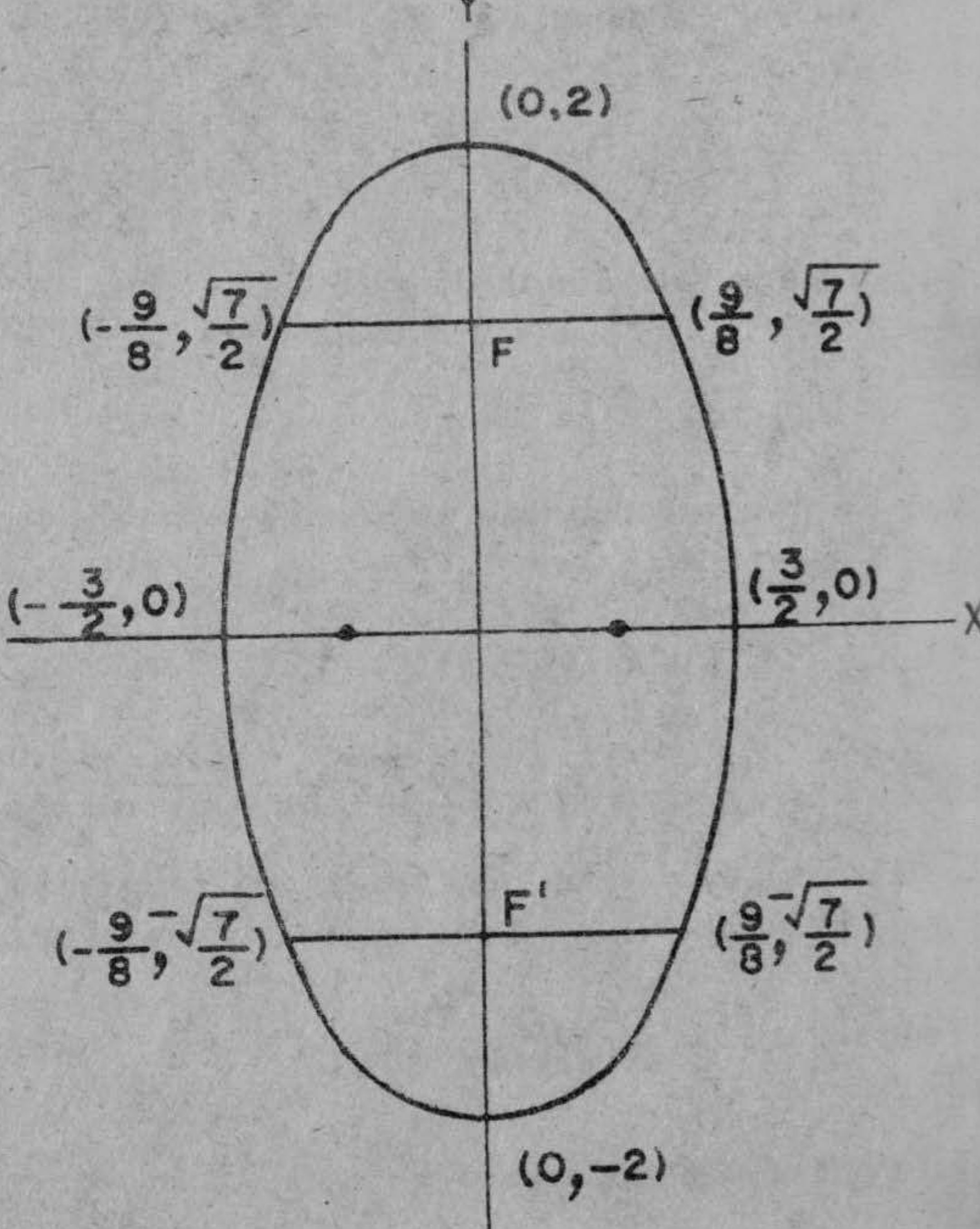

Y
(0,2)
$(-\frac{9}{8}, \frac{\sqrt{7}}{2})$
F
$(\frac{9}{8}, \frac{\sqrt{7}}{2})$
$(-\frac{3}{2},0)$
$(\frac{3}{2},0)$
X
F'
$(-\frac{9}{8}, -\frac{\sqrt{7}}{2})$
$(\frac{9}{8}, -\frac{\sqrt{7}}{2})$
(0,−2)

Fig 8.15

Exercise 8.6.

1. Find the semi-axes, the coordinates of the foci of the following ellipses and draw their graphs. :

(*i*) $4x^2+9y^2=36$
(*ii*) $25x^2+9y^2=225$
(*iii*) $25x^2+9y^2-25=0$
(*iv*) $x^2+25y^2-25=0$
(*v*) $4x^2+y^2=1$.

2. Find the equations of the following ellipses, whose centres are at the origin and which satisfy the given conditions :

(*i*) foci at $(\pm 4,0)$; vertices at $(\pm 5\ 0)$
(*ii*) foci at (0 ± 2); vertices at $(0, \pm 4)$
(*iii*) foci at $(\pm 3,0)$; length of minor axis $=6$
(*iv*) end points of minor axis at $(0, \pm 3)$, distance between foci $=8$.
(*v*) vertices at $(\pm 4,0)$; latus rectum of length 2
(*vi*) vertices at $(\pm 5,0)$; ends of minor axis $(0, \pm 1)$
(*vii*) ends of minor axis $(\pm 2,0)$, foci $(0, \pm 2)$
(*viii*) ends of minor axis $(0, \pm 3)$; latus rectum of length 3
(*ix*) foci $(0, \pm 4)$, latus rectum of length 12
(*x*) vertices $(0, \pm 5)$, passing through the point $(\frac{4}{5},3)$.

3 Use the definition of the ellipse to find the equation of the ellipse having foci $(3,5)$ and $(9,5)$ and having $a=5$.

4. Use the definition of the ellipse to find the equation of the ellipse having foci $(-1,-2)$ and $(-1,0)$ and having one vertex $(-1,1)$.

5. Find the equation of the circle passing through the focus of the parabola $x^2+8y=0$ and the foci of the ellipse $16x^2+25y^2-400=0$.

6. Prove that the line $lx+my+n=0$ and the ellips $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ have just one point in common if $a^2l^2+b^2m^2=n^2$,

ANSWERS.

1. (i) foci $(\pm\sqrt{5},0)$; 3,2
   (ii) foci $(0,\pm 4)$; 3,5
   (iii) foci $(0,\pm\frac{4}{3})$; $1,\frac{5}{3}$
   (iv) foci $(\pm 2\sqrt{6},0)$; 5,1
   (v) foci $\left(0,\pm\frac{\sqrt{3}}{2}\right)$; $\frac{1}{2}$, 1.
2. (i) $9x^2+25y^2=225$.
   (ii) $\frac{x^2}{12}+\frac{y^2}{16}=1$
   (iii) $x^2+2y^2-18=0$
   (iv) $9x^2+25y^2-225=0$
   (v) $x^2+4y^2=16$
   (vi) $x^2+25y^2=25$
   (vii) $2x^2+y^2=8$
   (viii) $x^2+4y^2=36$
   (ix) $4x^2+3y^2=192$
   (x) $y^2+25x^2=25$
3. $16x^2+25y^2-192x-250y+801=0$
4. $4x^2+3y^2+8x+6y-5=0$
5. $2x^2+2y^2-5y-18=0$

8.12. *A Focus-Directrix Property of the Ellipse*

Using the notation of the § 8.11, the ratio $\frac{c}{a}$ is called the eccentricity of the ellipse. The eccentricity is usually denoted by e

$$\text{Thus we have } \frac{c}{a} = e \text{ or } c = ae \qquad (1).$$

Since $c < a$, we must have $e < 1$. From relation (1) of § 8.11 it follows that

$$a^2 = b^2 + a^2e^2,$$

$$\text{or } a^2(1 - e^2) = b^2 \qquad (2).$$

From (2) it is apparent that if $e = 0$, then $a = b$ and the ellipse is a circle. Henceforth we shall assume that $e > 0$.

If e is very small, then a and b will be almost equal and the ellipse will be nearly circular. On the other hand if e is almost equal to 1, so that $1 - e^2$ is very small, b will be small in comparison with a and the ellipse will be long and narrow Thus the shape of an ellipse depends upon the value of its eccentricity.

We now define the directrices of an ellipse to be the lines perpendicular to the major axis at a distance of $\frac{a}{e}$ from the centre of the ellipse on both sides of the centre. Since $e < 1$, we have $\frac{a}{e} > a$, and so the directrices do not intersect the ellipse.

If an ellipse has its centre at the origin and its major axis along the x-axis, then its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the directrices have equations $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ and the foci have coordinates $(ae, 0)$ and $(-ae, 0)$, Figure 8.16.

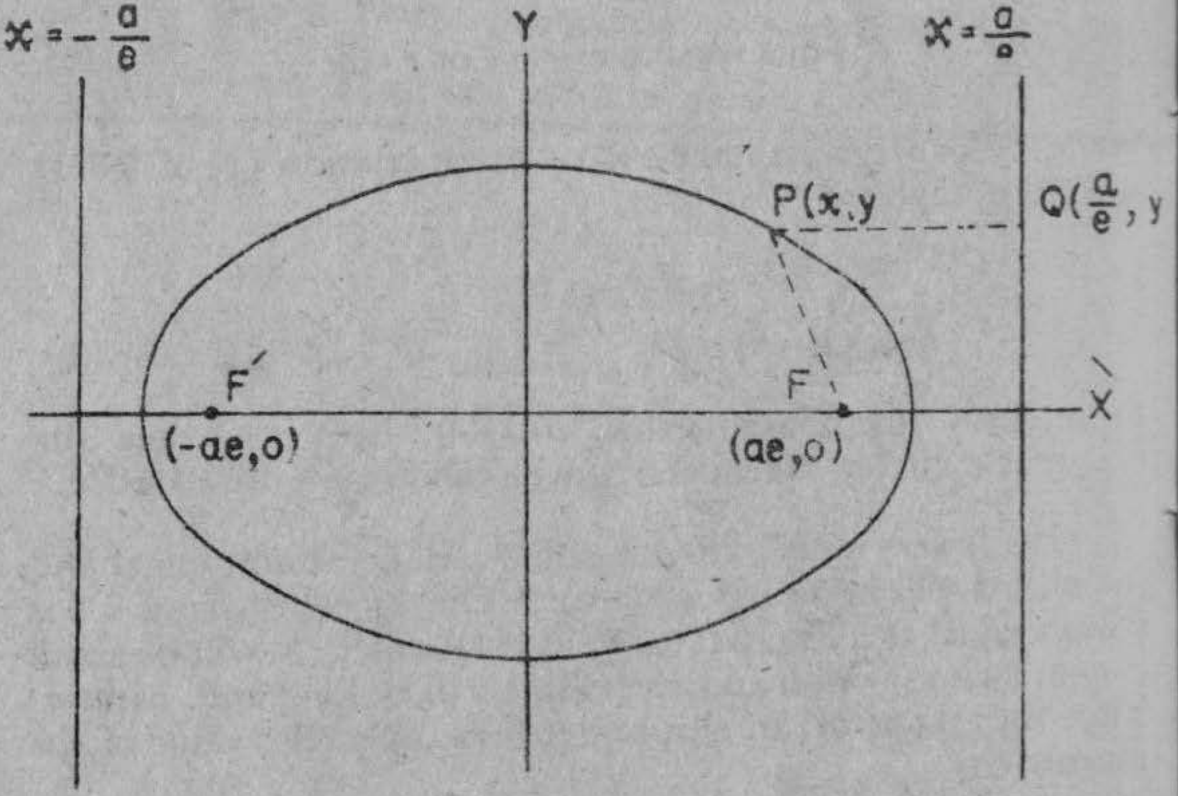


Fig. 8.16

We now prove the following theorem.

8.12.1. *Theorem.* For any point on an ellipse the ratio of its distance from a focus to its distance from the corresponding directrix is constant and is equal to the eccentricity.

Proof. Let P(x, y) be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \ldots\ldots(1)$$

where $b^2=a^2(1-e^2)$.

If F $(ae, 0)$ is one focus, the corresponding directrix is

$$x=\frac{a}{e}.$$

Now distance of P from the focus F $=\sqrt{(x-ae)^2+y^2}$ and distance of P from the directrix is

$$\frac{a}{e}-x.$$

Hence $$\frac{\sqrt{\left(x-ae\right)^2+y^2}}{\frac{a}{e}-x}=\frac{\sqrt{(x-ae)^2+b^2\left(1-\frac{x^2}{a^2}\right)}}{\frac{a}{e}-x}$$

$$=\frac{\sqrt{(x-ae)^2+a^2(1-e^2)\left(1-\frac{x^2}{a^2}\right)}}{\frac{a}{e}-x}$$

$$=\frac{\sqrt{x^2-2xae+a^2e^2+a^2(1-e^2)-(1-e^2)x^2}}{\frac{a-ex}{e}}$$

$$=\frac{\sqrt{e^2x^2-2xae+a^2}}{\frac{a-ex}{e}}=\frac{a-ex}{\frac{a-ex}{e}}=e$$

which proves the result.

Example. 1 Find the equation of the ellipse having the origin as its centre, one focus at the point (4, 0) and the corresponding directrix $x=6$.

Solution. Since $c=4$ for this ellipse we have $ae=4$

By definition of the directrices, it follows that $\frac{a}{e}=6$.

Multiplying corresponding members of these two equations, we find that, $a^2=24$

Thus $b^2=24-16=8$

Since the major axis is along the x – **axis**, the required equation is

$$\frac{x^2}{24}+\frac{y^2}{8}=1.$$

Example 2. Find the eccentricity, the coordinates of the foci, and the equation of the directrices of the ellipse

$$\frac{x^2}{9}+\frac{y^2}{25}=1.$$

Solution. For this ellipse we have $a=5$, $b=3$.

Thus $c=4$. Hence $ae=4$ and $e=\frac{4}{5}$. Since the major axis is along the y-axis, the coordinates of the foci are (0, 4) and (0, −4).

$$\text{Now}\ \frac{a}{e}=\frac{25}{4},$$

so the equations of the directrices are

$$y=\frac{25}{4}\ \text{and}\ y=-\frac{25}{4}.$$

Exercise 8.7

1. Find the eccentricity, foci, and directrices of each of the following ellipses :

(*i*) $16x^2+25y^2=400$

(*ii*) $4x^2+3y^2=12$

(*iii*) $20x^2+36y^2=5$

(*iv*) $9x^2+y^2=1$.

2. Find the equation of the ellipse with centre at the origin and satisfying the given conditions :

(*i*) focus (2, 0), directrix $x=8$

(*ii*) focus (0, −3), directrix $y=-4$

(*iii*) eccentricity $\frac{1}{3}$, directrix $x=12$

(*iv*) eccentricity $\frac{3}{5}$, end of minor axis (0 − 8)

(*v*) directrix $x=16$, length of latus rectum 12.

(*vi*) eccentricity $\frac{2}{3}$, length of latus rectum $\frac{20}{3}$, major axis along the y-axis.

ANSWERS

(i) $\frac{3}{5}$; $(\pm 3, 0)$; $x = \pm \frac{25}{3}$

(ii) $\frac{1}{2}$; $(0, \pm 1)$; $y = \pm 4$

(iii) $\frac{2}{3}$; $(\pm \frac{1}{3}, 0)$; $x = \pm \frac{3}{4}$

(iv) $\frac{2\sqrt{2}}{3}$; $\left(0 \pm \frac{2\sqrt{2}}{3}\right)$; $y = \pm \frac{3\sqrt{2}}{4}$

2. (i) $3x^2 + 4y^2 = 48$

(ii) $4x^2 + y^2 = 12$

(iii) $8x^2 + 9y^2 = 128$

(iv) $\frac{x^2}{100} + \frac{y^2}{64} = 1$

(v) $\frac{x^2}{64} + \frac{y^2}{48} = 1$

or $\frac{x^2}{32(7-\sqrt{13})} + \frac{y^2}{24(\sqrt{13}-1)} = 1$

(vi) $\frac{x^2}{20} + \frac{y^2}{36} = 1.$

8.13. *A More General form of the Equation of the Ellipse*

The equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \; a > b,$$

represents an ellipse which has centre (h,k), equation of minor axis is $x - h = 0$ and that of major axis is $y - k = 0$ Semi-major and semi minor axes are a and b respectively

In general,

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

represents an ellipse if $a \neq 0$, $b \neq 0$ and a and b have the same sign

Example Find the coordinates of the centre and the foci, the semi-axes and the eccentricity of the ellipse

$$9x^2+25y^2-54x+50y-119=0.$$

Solution. $9x^2+25y^2-54x+50y-119=0$

$$\Rightarrow \quad 9\,(x^2-6x)+25\,(y^2+2y)=119$$

$$\Rightarrow \quad 9\,(x^2-6x+9)+25\,(y^2+2y+1)=119+81+25$$

$$\Rightarrow \quad 9\,(x-3)^2+25\,(y+1)^2=225$$

$$\Rightarrow \quad \frac{(x-3)^2}{25}+\frac{(y+1)^2}{9}=1.$$

Thus the centre of the ellipse is the point $(3,-1)$ and the semi-axes are $a=5$ and $b=3$.

Vertices are $(8,-1)$ and $(-2,-1)$ and the extremities of the minor axis are $(3, 2)$ and $(3, -4)$.

Since $c=4$, the foci are $(-1,-1)$, $(7,-1)$ and the eccentricity is $e=\frac{4}{5}$.

Exercise 8.8.

Find the coordinates of the centre and the foci, the semi-axes and eccentricity of the following ellipses. Also sketch their graphs.

1. $9x^2+25y^2-36x+50y-164=0$
2. $4x^2+y^2+16x-6y-39=0$
3. $x^2+9y^2-54y+72=0$
4. $2x^2+18y^2-6x+18y-9=0.$

ANSWERS

1. C : $(2,-1)$; F : $(-2,-1)$ and $(6,-1)$; $a=5$. $b=3$; $e=\frac{4}{5}$.

2. $C : (-2, 3)$; $F : (-2, 3 \pm 4\sqrt{3})$; $a=8$, $b=4$; $e=\frac{1}{2}\sqrt{3}$.
3. $C : (0, 3)$; $F : (\pm 2\sqrt{2}, 3)$; $a=3$, $b=1$; $e=\frac{2}{3}\sqrt{2}$.
4. $C : (\frac{3}{2}, -\frac{1}{2})$; $F : \frac{3}{2} \pm 2\sqrt{2}, -\frac{1}{2})$; $a=3$, $b=1$; $e=\frac{2}{3}\sqrt{2}$.

8.14. *Equation of the Hyperbola*

Let F and F′ be two distinct fixed points and $2c$ the distance between them. Suppose a is a real number such that $0<a<c$. Then the locus of all points P such that $FP| - |F'P| = \pm 2a$ is called a *hyperbola*.

The two fixed points, F and F′ are called the *foci* of the hyperbola, and the mid. point O of the segment FF′ is called the *centre* of the hyperbola. Since $|FF'| = 2c$,

$$\therefore \quad |OF| = |OF'| = c.$$

To derive the equation of the hyperbola, let the line

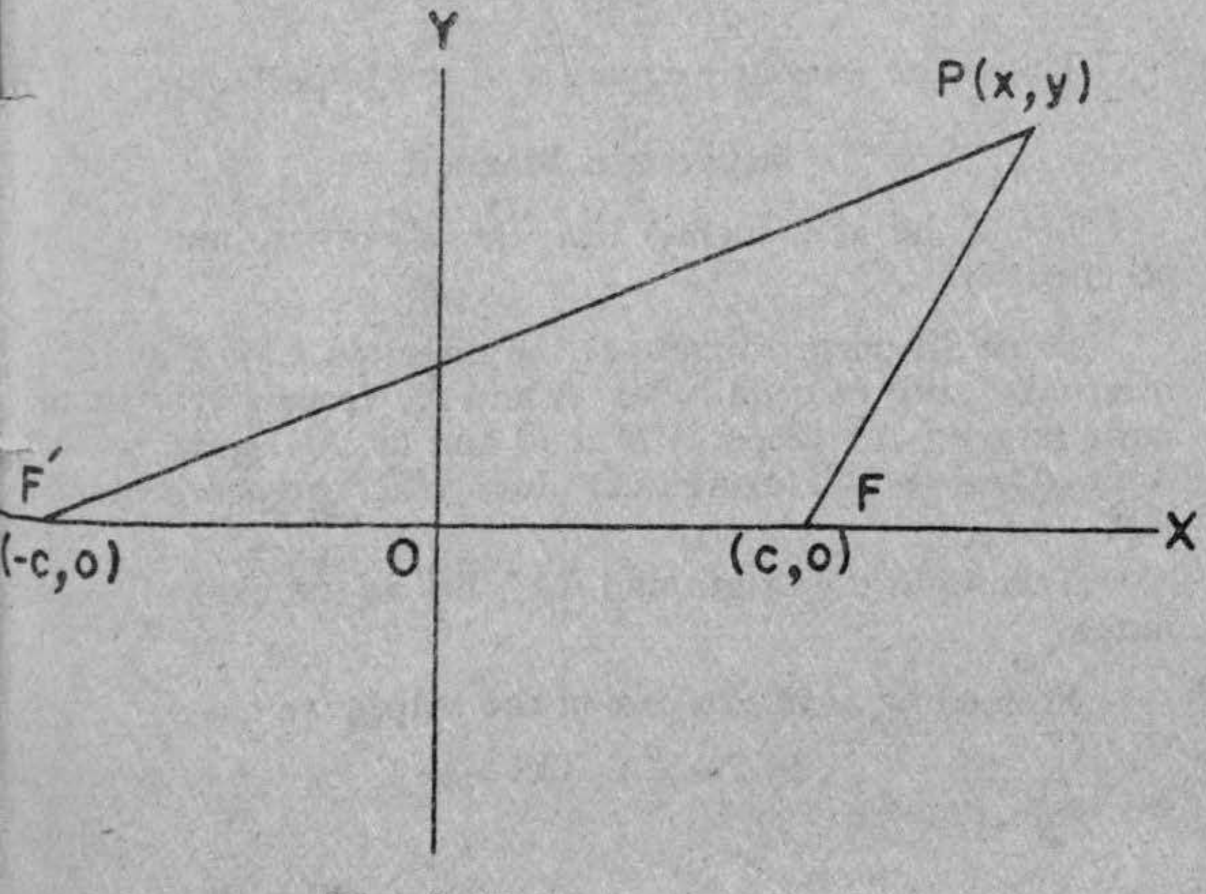


Fig.8.17.1

which joins the two foci F′ and F be the x-axis and the centre of the hyperbola be the origin.

Then the foci have coordinates $(-c, 0)$ and $(c, 0)$. Now a point P (x, y) is on the hyperbola if

$$|FP| - |F'P| = \pm 2a$$

that is, if

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a.$$

This equation is similar in form to the equation of the ellipse. Hence a similar reduction gives the result.

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1,$$

since $a<c$, $a^2-c^2<0$ and so $c^2-a^2>0$.

Thus we may write

$$b^2 = c^2 - a^2. \tag{1}$$

Hence the equation of a hyperbola can be written in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This is the standard equation of the hyperbola.

Alternate Method

Let S be the focus, LM the directrix and e the eccentricity.

Draw SZ perpendicular to the directrix LM. Divide SZ internally and externally at A and A′ respectively in the same ratio $e : 1$. Bisect A′A at O and let $A'A = 2a$, so that $A'O = OA = a$ Through O draw BB′ perpendicular to AA′.

Take O as the origin and AA′, BB′ as the axes of coordinates.

Proceeding as in the case of the ellipse we have

$$OZ = \frac{a}{e}, \quad OS = ae.$$

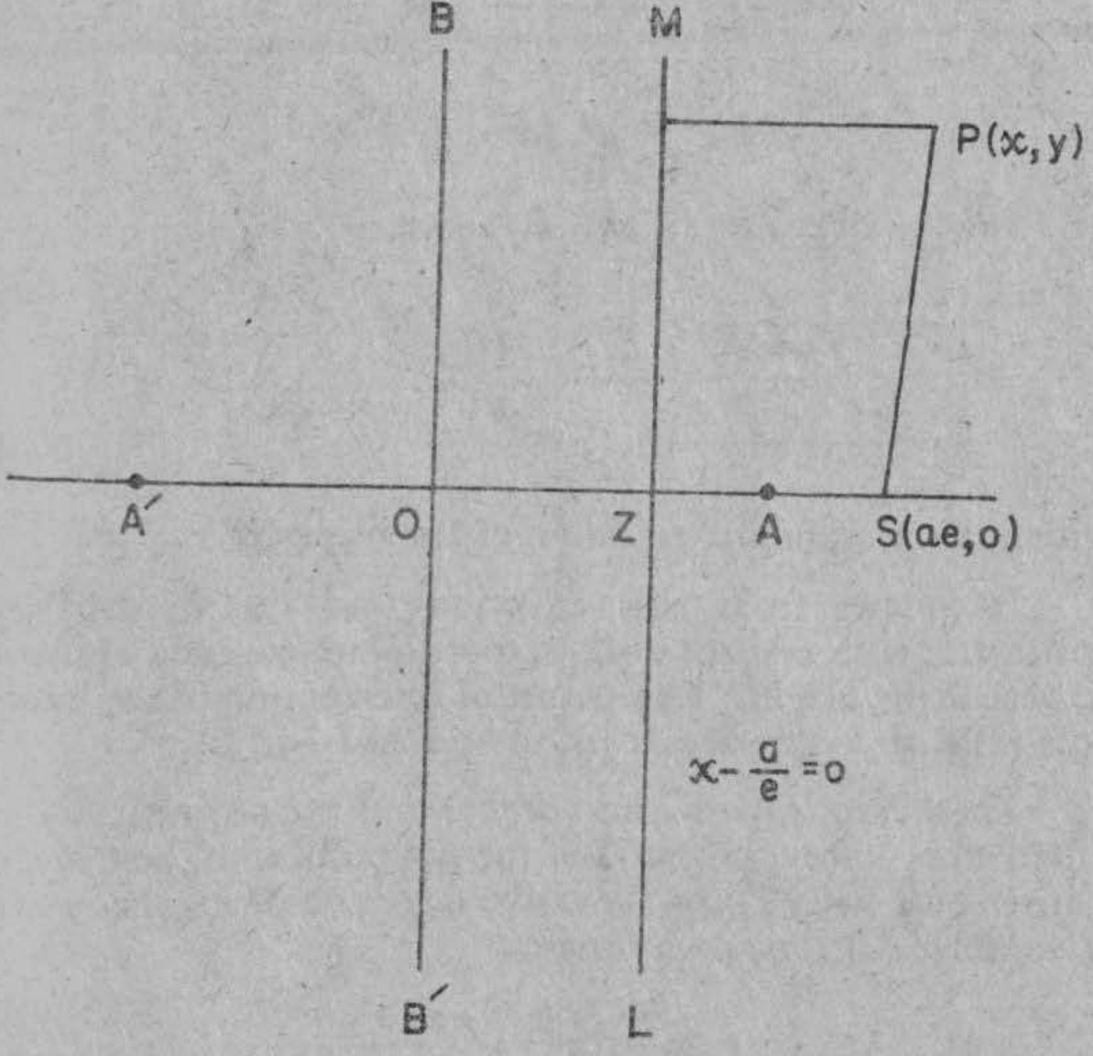

B
M
P(x, y)
A′
O
Z
A
S(ae, o)
$x-\frac{a}{e}=o$
B′
L

Fig.8.17.2

Thus the coordinates of S and Z are $(de, 0)$ and $\left(\frac{a}{e}, 0\right)$ respectively and hence the equation of the directrix LM is

$$x=\frac{a}{e} \text{ or } x-\frac{a}{e}=0.$$

Let P (x, y) be a point on the hyperbola then by definition

PS $=(e)$ (length of perpendicular from P to LM

or $\sqrt{(x-ae)^2+y^2}=e\left(x-\frac{a}{e}\right)$,

or $(x-ae)^2+y^2=e^2\left(x-\frac{a}{e}\right)^2$.

which (after simplification) reduces to

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1,$$

$$\text{or } \frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad \text{.........(2)}$$

which is the standard equation of the hyperbola.

It follows from this equation that the hyperbola is symmetric with respect to both coordinate axes and also with respect to the origin. The points of intersection of the hyperbola (2) with x-axis are A $(a, 0)$ and A $(-a, 0)$.

These are called the vertices and the segment AA is called the transverse axis of the hyperbola. There are no y-intercepts (why ?), so the curve does not meet the y-axis. If we solve (2) for y, we obtain

$$y=\pm\frac{b}{a}\sqrt{x^2-a^2}\,.$$

Real values of y occur if. $x^2\geqslant a^2$ that is, if $x\geqslant a$ or $x\leqslant -a$. Accordingly the hyperbola consists of two branches. one in the region where $x\geqslant a$ and the other in the region where $x\leqslant -a$.

The general shape of the hyperbola is indicated in figure 8.18.

A latus rectum of a hyperbola is a chord which passes through a focus and is perpendicular to the transverse axis. Since there are two foci of hyperbola, it will have two latera recta. In figure 8.18. CD and C′ D′ are the latera recta.

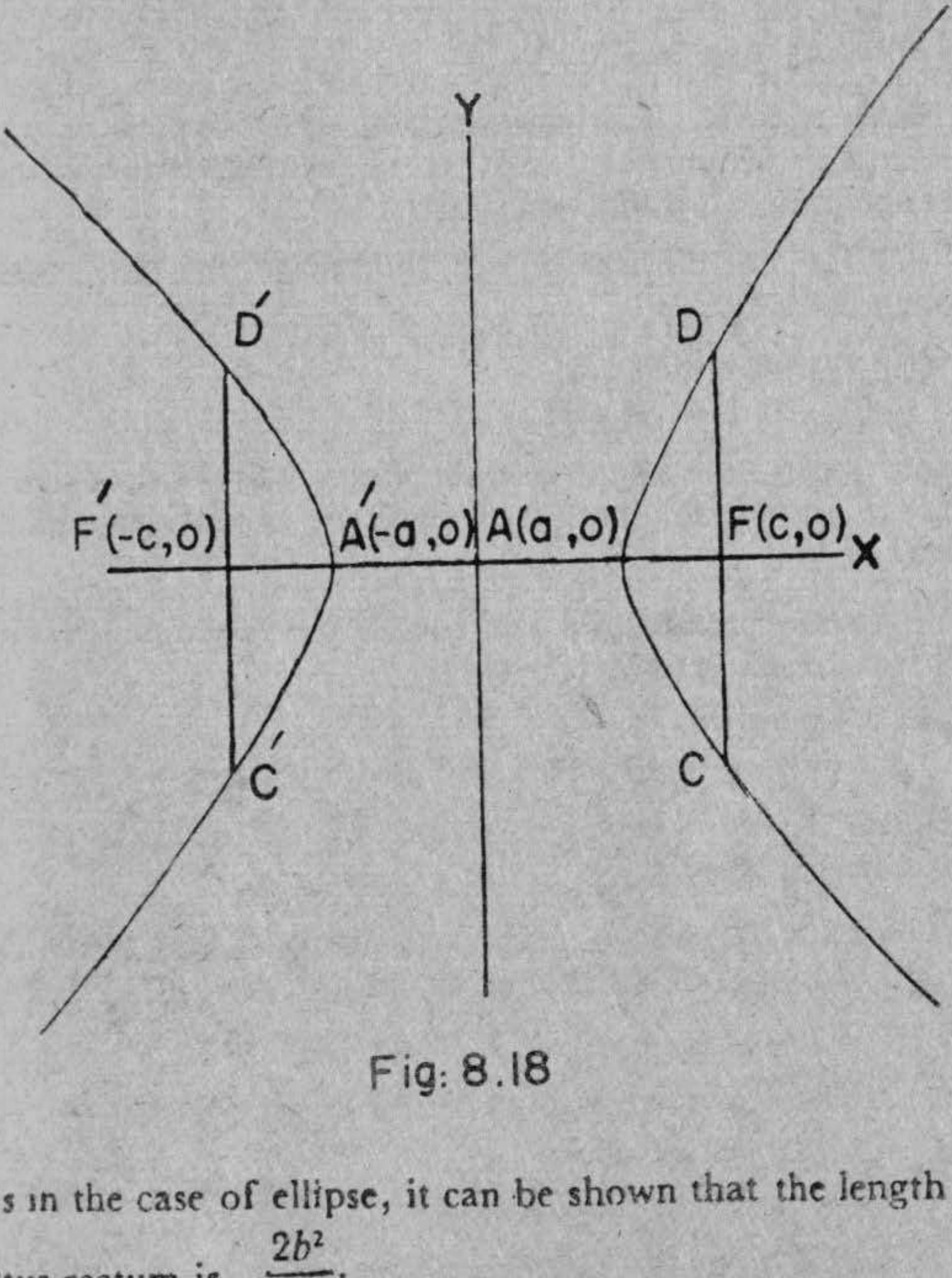


Fig: 8.18

As in the case of ellipse, it can be shown that the length of a latus rectum is $\frac{2b^2}{a}$.

If the y-axis, instead of the x-axis, is chosen as the line through the foci, and the origin as the centre of the hyperbola, the equation becomes

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \qquad \ldots\ldots\ldots(3)$$

In this case $(0, a)$ and $(\theta - a)$ are the vertices, and (0,

$(0, -c)$ are the foci of the hyperbola. Clearly there are no x-intercepts in this case.

Note. It is clear that ellipse and hyperbola have many similar properties. But there are several points of difference between them, we have :

(*i*) for an ellipse $a > c$, whereas in the case of a hyperbola $a < c$,

(*ii*) for an ellipse

$$b^2 = a^2 - c^2,$$

therefore c can take any value between 0 and a but cannot exceed a in which case b^2 will be negative which is impossible.

When $c = 0$ then $b = a$ and the ellipse becomes a circle whose foci coincide at the centre.

For a hyperbola

$$b^2 = c^2 - a^2.$$

Therefore c cannot be zero since $c = 0$ implies $b^2 = -a^2$, which is impossible.

Hence a hyperbola has two distinct foci. We will notice other points of difference as we proceed on.

8.15. *The Asymptotes of a Hyperbola*

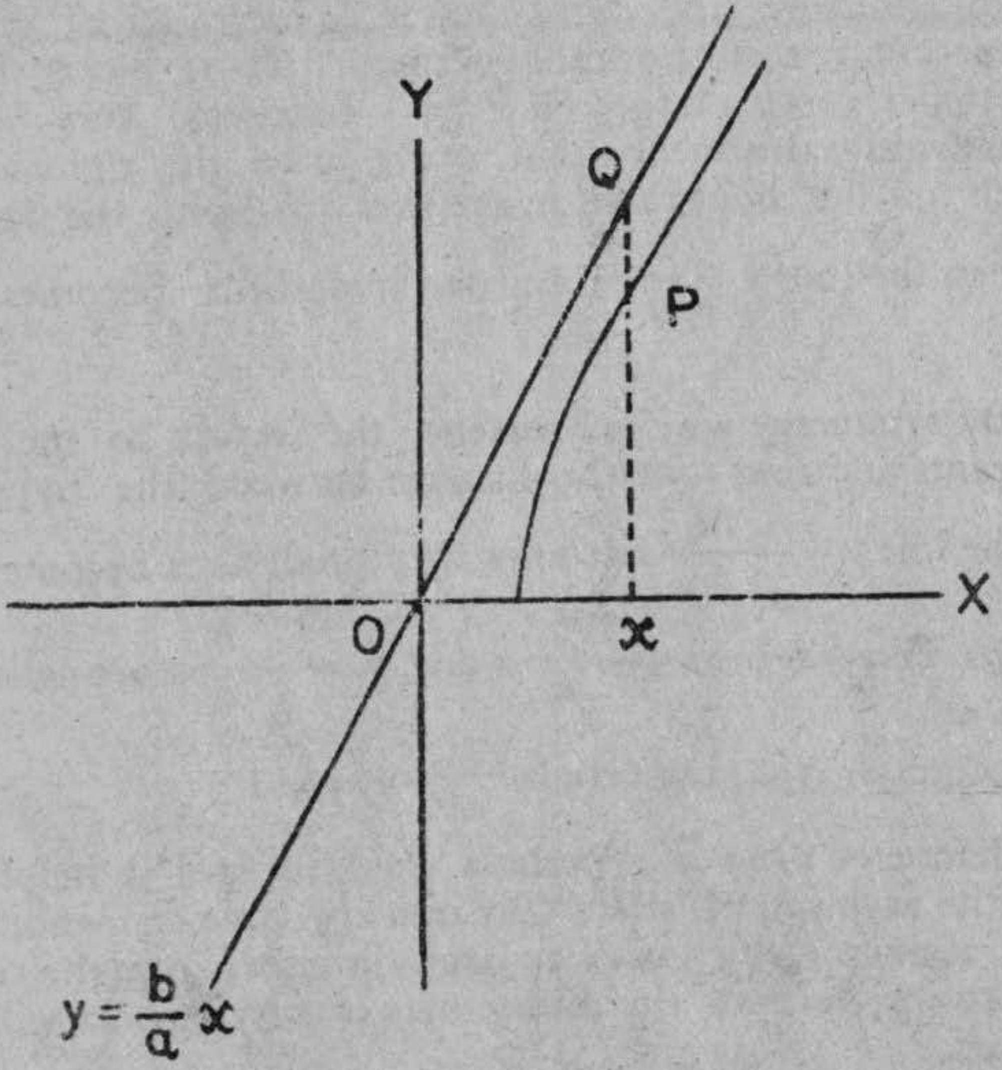
Let us consider that part of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ which is in the first quadrant.}$$

For any $x > a$, the corresponding value of y is given by

$$y = \frac{b}{a}\sqrt{x^2 - a^2}.$$

Since $x^2 - a^2 < x^2$ and therefore $\sqrt{x^2 - a^2} < x$, we see that $y < \frac{b}{a}x$. This means that in the first quadrant the hyperbola lies below the line $y = \frac{b}{a}x$, We now show that as we go to the right on the hyperbola, the hyperbola approaches nearer and nearer to the line. Let P and Q denote the respective points on the hyperbola and the line with the same x-coordinate 'x'.



Figur 8.19

Then $|PQ|$ is equal to the y-coordinate of Q minus the y-coordinate of P, that is

$$|PQ| = \frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}$$

$$=\frac{b}{a}(x-\sqrt{x^2-a^2})$$

$$=\frac{ab}{x+\sqrt{x^2-a^2}}.$$

Now if x is large, the denominator is large ($>x$) so tha the fraction is correspondingly small. It is clear that a x becomes very large, $|PQ|$ becomes very small. Geometrically this means that, as we go to the right on that branch of the hyperbola in the first quadrant, the distanc between the line $y=\frac{b}{a}x$ and the hyperbola becomes ver small.

By symmetry we can extend the result to the other quadrants and find that the distance between the hyperbola and the line $y=-\frac{b}{a}x$ becomes very small as x becomes very large. The two lines $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$ are called the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Before we draw a hyperbola we will find it helpful to draw the asymptotes, since they quickly give an indication of the position of the curve at large distances from the centre. An easy way to draw the asymptotes of a hyperbola will now be given.

Construct the rectangle whose sides are

$$x=\pm a, \quad y=\pm b$$

(Figure 8.20). The diagonals of this rectangle have slopes $\frac{b}{a}$ and $-\frac{b}{a}$, and each passes through the origin (the centre of the rectangle). Thus the asymptotes of the hyperbola are the diagonals of this rectangle.

For the hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ having its foci on the y-axis, the asymptotes are the lines $y = \pm \frac{a}{b}x$. They are the diagonals of the rectangle constructed by the lines $x = \pm b$, $y = \pm a$.

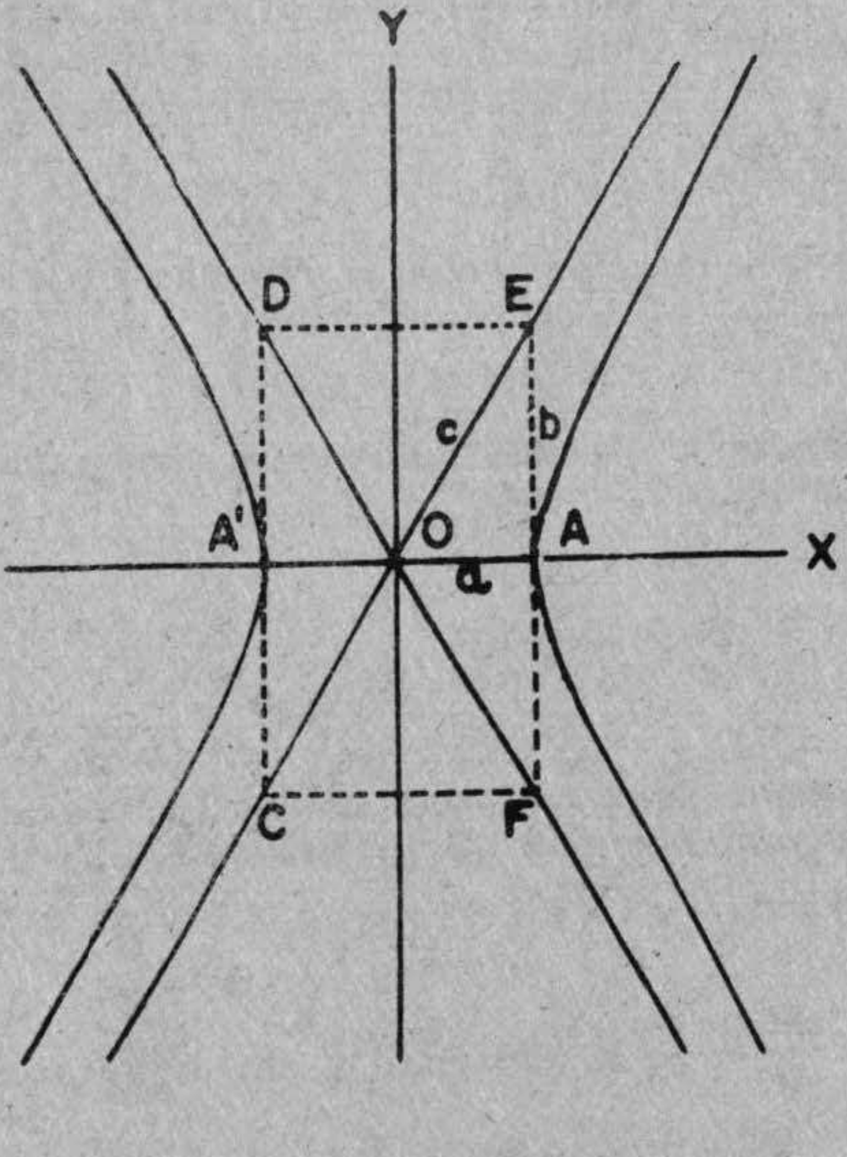


Figure 8.20

If the two asymptotes of a hyperbola are perpendicular, the hyperbola is called a rectangular hyperbola.

A hyperbola with equation (2), is therefore a rectangular hyperbola if

$$\left(\frac{b}{a}\right)\left(-\frac{b}{a}\right)=-1$$

that is if $a=b$

The same conclusion holds in case the equation is of the form (3).

Example 1. Discuss and draw the graph of the hyperbola

$$\frac{x^2}{4}-\frac{y^2}{9}=1.$$

Solution. Here $a=2$ and $b=3$. Therefore $c=\sqrt{13}$. The vertices are therefore (2, 0) and (−2, 0). Foci are $(\sqrt{13, 0})$ and $(-\sqrt{13, 0})$.

Now $\frac{b^2}{a}=\frac{9}{2}$, so the ends of the latera recta have coordinates $\left(\pm\sqrt{13}, \pm\frac{9}{2}\right)$. The asymptotes are the lines $y=\frac{3x}{2}$ and $y=-\frac{3x}{2}$. The graph is drawn in figure 8.21.

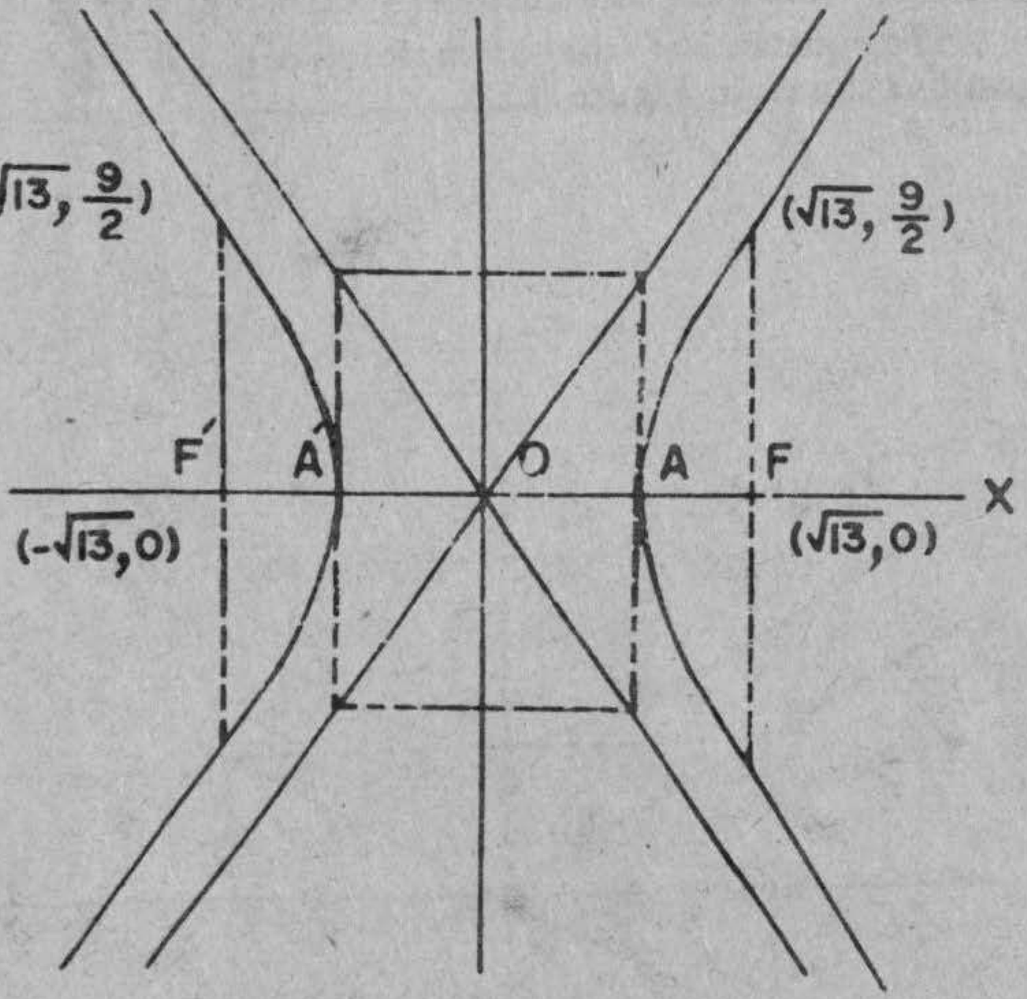

$(-\sqrt{13}, \frac{9}{2})$
$(\sqrt{13}, \frac{9}{2})$
F′
A′
O
A
F
X
$(-\sqrt{13}, 0)$
$(\sqrt{13}, 0)$

Fig. 8.21

Example 2. Discuss and draw the graph of the hyperbola $16y^2 - 9x^2 = 144$.

Solution. The given equation can be written in the form

$$\frac{y^2}{9} - \frac{x^2}{16} = 1,$$

Here $a=3$, $b=4$ and $c=5$. Since this equation is of the form (3), the foci are on the y-axis and therefore have coordinates $(0, \pm 5)$. The vertices are $(0, \pm 3)$ and the ends of the latera recta have coordinates $(\pm \frac{16}{5}, \pm 5)$.

The equations of the asymptotes are $y = \pm \frac{3}{4} x$. The graph is drawn in Figure 8.22.

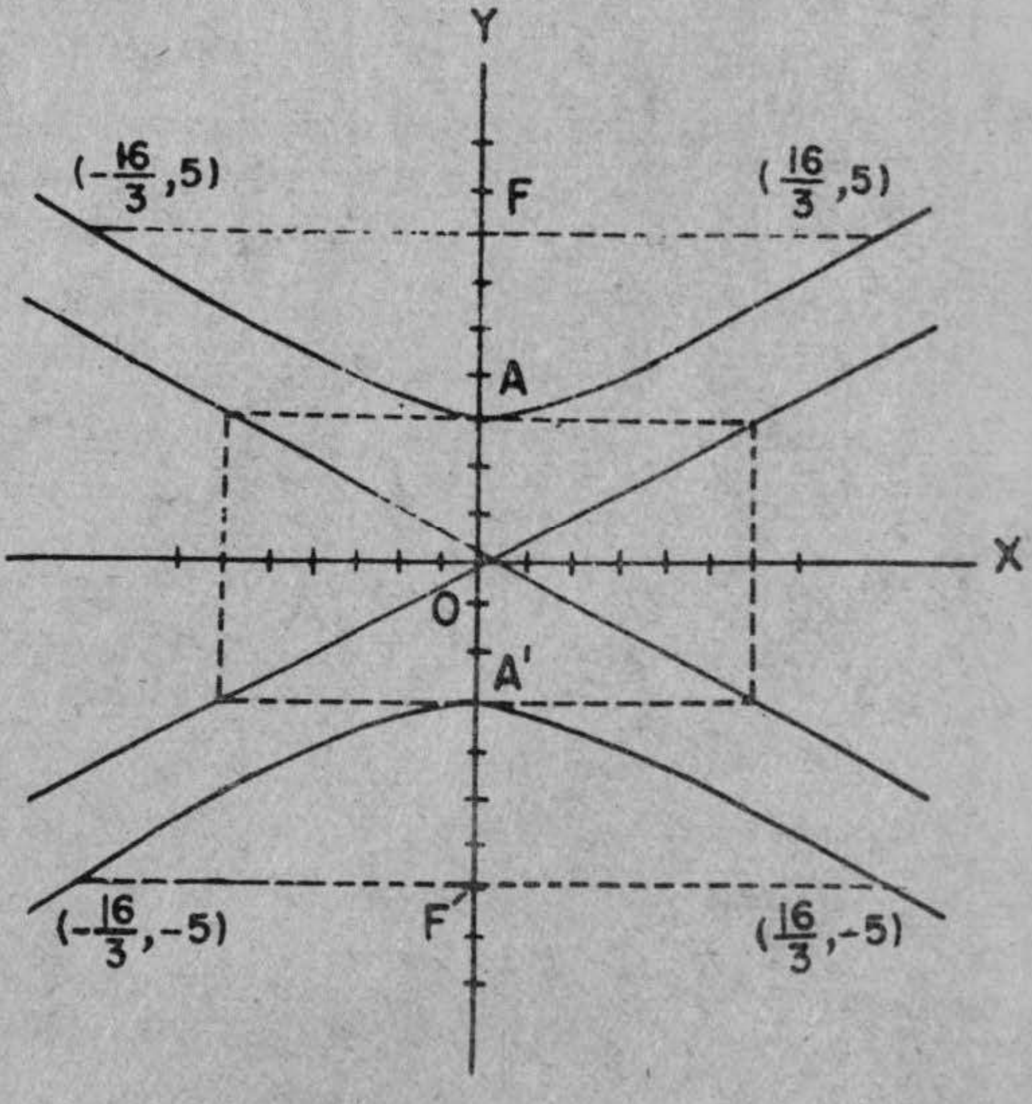

Y
$(-\frac{16}{3}, 5)$
F
$(\frac{16}{3}, 5)$
A
X
O
A'
$(-\frac{16}{3}, -5)$
F'
$(\frac{16}{3}, -5)$

Fig. 8.22

Exercise 8.9.

1. Discuss and draw the graph of the following equations.

(*i*) $y^2-x^2=4$

(*ii*) $25y^2-144x^2=3600$

(*iii*) $4x^2-9y^2=36$

(*iv*) $4x^2-3y^2-12=0$

(*v*) $x^2-4y^2+4=0$

(*vi*) $x^2-y^2-1=0$.

2. Find the equation of the hyperbola satisfying the following conditions :

(*i*) foci $(\pm 5, 0)$, vertices $(\pm 3, 0)$

(*ii*) vertices $(\pm 4, 0)$, latus rectum of length 2

(*iii*) foci $(0, \pm 3)$, vertices $(0, \pm 2)$

(*iv*) vertices $(0, \pm 3)$, rectangular hyperbola

(*v*) foci $(\pm\sqrt{13}, 0)$, asymptotes $y=\pm\frac{2}{3}x$

(*vi*) vertices $(0, \pm 4)$, asymptotes $y=\pm 3x$

(*vii*) asymptotes $y=\pm x$, latus rectum of length 10

(*viii*) vertices $(\pm 1, 0)$, passing through the point $(-3,8)$.

3. Use the definition of the hyperbola to find the equation of the hyperbola having foci $(-1, 2)$ and $(9, 2)$ and one vertex $(7, 2)$.

4. Use the definition of hyperbola to find the equation of the hyperbola having foci $(3. \pm 4)$ and vertices $(3, \pm 3)$.

5. Prove that a line parallel to an asymptote intersects the hyperbola in just one point.

6. Prove that the products of the distances of any point on a hyperbola from its asymptotes is constant.

ANSWERS

1. (*i*) Vertices $(0, \pm 2)$; foci $(0, \pm 2\sqrt{2})$; asymptotes $y = \pm x$

    (*ii*) Vertices $(0, \pm 12)$; foci $(0, \pm 13)$; asymptotes $5y \pm 12x = 0$

    (*iii*) Vertices $(\pm 3, 0)$; foci $(\pm \sqrt{13}, 0)$; asymptotes $3y \pm 2x = 0$

    (*iv*) Vertices $(\pm \sqrt{3}, 0)$; foci $(\pm \sqrt{7}, 0)$; asymptotes $3y \pm 2\sqrt{3}\, x = 0$

    (*v*) Vertices $(0, \pm 1)$; foci $(0, \pm \sqrt{5})$; asymptotes $2y \pm x = 0$

    (*vi*) Vertices $(\pm 1, 0)$; foci $(\pm \sqrt{2}, 0)$; asymptotes $y \pm x = 0$.

2. (*i*) $16x^2 - 9y^2 = 144$

    (*ii*) $x^2 - 4y^2 = 16$

    (*iii*) $5y^2 - 4x^2 = 20$

    (*iv*) $y^2 - x^2 = 9$

    (*v*) $4x^2 - 9y^2 = 36$

    (*vi*) $y^2 - 9x^2 = 16$

    (*vii*) $x^2 - y^2 = +25$

(*viii*) $8x^2-y^2-8$

3. $16x^2-9y^2-128x+36y+76=0$

4. $9x^2-7y^2-54x+144=0.$

8.16. *Focus Directrix Property of the Hyperbola*

As in the case of ellipse, the ratio $\frac{c}{a}$ is called the eccentricity of a hyperbola. If eccentricity is denoted by e, we have $\frac{c}{a}=e$ or $c=ae$.

The directrices of a hyperbola are the lines perpendicular to the transverse axis at a distance $\frac{a}{e}$ from the centre of the hyperbola.

Since for a hyperbola $c>a$, $e>1$. It follows that $\frac{a}{e}<a$ and therefore the directrices of a hyperbola do not intersect the hyperbola. Since $b^2=c^2-a^2$, we have

$$b^2=a^2(e^2-1) \tag{1}$$

If the hyperbola has its centre at the origin and its transverse axis along the x-axis, its equation is of the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (2).$$

In this case the foci are the points F $(ae,0)$ and F′ $(-ae,0)$, and the directrices are the lines $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ (Figure 8.23).

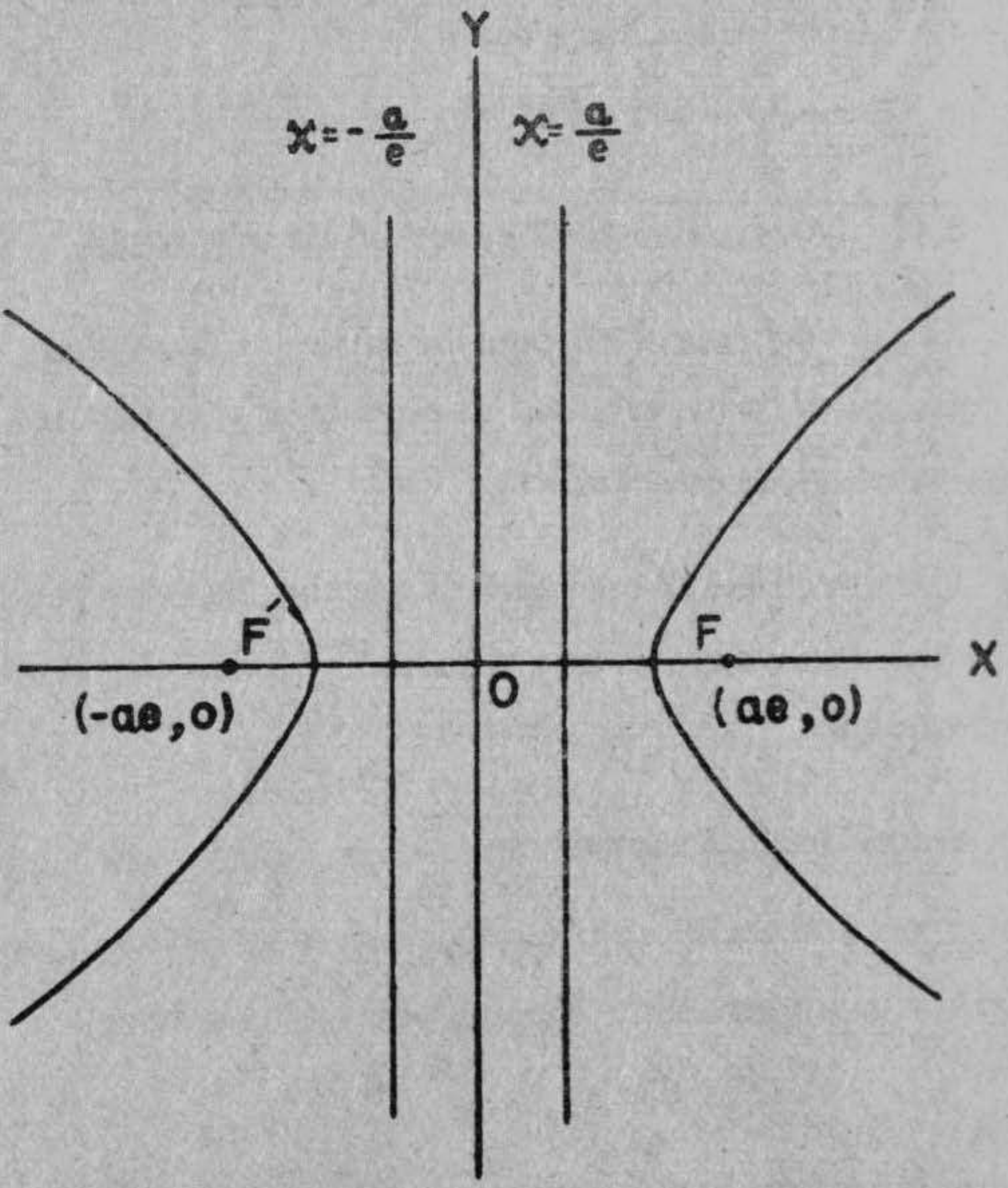

Y
$x = -\frac{a}{e}$
$x = \frac{a}{e}$
F'
F
(-ae, o)
O
(ae, o)
X

Fig. 8.23

8.16.1. *Theorem*. For any point on a hyperbola the ratio of its distance from a focus to its distance from the corresponding directrix is constant and is equal to the eccentricity

Proof : The proof is simiiar to the proof of the Theorem 8.12.1 for an ellipse.

Example. Find the eccentricity, the coordinates of foci and the equations of the directrices of the hyperbola

$$\frac{x^2}{4}-\frac{y^2}{16}=1.$$

Solution. Here $a=2$, $b=4$

therefore $c=2\sqrt{5}$

Since the transverse axis is along the x—axis, the foci have coordinates $(2\sqrt{5},0)$ and $(-2\sqrt{5},0)$. From $e=ae$ we find $e=\sqrt{5}.$ The equations of the directrices are $x=\frac{2}{\sqrt{5}}$ and $x=-\frac{2}{\sqrt{5}}.$

Exercise 8.10.

1. Find the eccentricity, foci and directrices of each of the following hyperbolas :—

(*i*) $16x^2-9y^2=144$

(*ii*) $x^2-y^2+4=0$

(*iii*) $9x^2-y^2-9=0$

(*iv*) $9x^2-y^2+1=0.$

2. Find the equation of the hyperbola with centre at the origin satisfying the conditions given in each of the following cases :

(*i*) focus (8,0), directrix $x=2$

(*ii*) focus (0,2), directrix $y=\frac{1}{18}$

(*iii*) eccentricity 2, focus (6,0)

(*iv*) eccentricity $\frac{3}{2}$, directrix $x=3$

(*v*) vertex (0,5), eccentricity 3

(*vi*) eccentricity $\frac{5}{3}$, length of latus rectum $\frac{64}{3}$, transverse axis along y — axis.

3. Prove that the eccentricity of any rectangular hyperbola is equal to $\sqrt{2}$.

4. Find the equations of the asymptotes of the hyperbola having its centre at the origin and eccentricity 3, if (*i*) the transverse axis is along the x—axis, (*ii*) the transverse axis is along the y—axis.

ANSWERS.

1. (*i*) $\frac{5}{3}$; $(\pm 5,0)$; $x=\pm\frac{9}{5}$
   (*ii*) $\sqrt{2}$; $(0,\pm 2\sqrt{2})$; $y=\pm\sqrt{2}$
   (*iii*) $\sqrt{10}$; $(\pm\sqrt{10},0)$; $x=\pm\frac{\sqrt{10}}{10}$
   (*iv*) $\frac{\sqrt{10}}{3}$; $\left(0,\pm\frac{\sqrt{10}}{3}\right)$, $y=\pm\frac{3}{\sqrt{10}}$
2. (*i*) $3x^2-y^2=48$
   (*ii*) $315y^2-9x^2=35$
   (*iii*) $3x^2-y^2=27$
   (*iv*) $20x^2-16y^2=405$
   (*v*) $8y^2-x^2=200$
   (*vi*) $16y^2-9x^2=576$

4. $y=\pm 2\sqrt{2}\,x$, $y=\pm\frac{1}{2\sqrt{2}}x$.

8.17. *A More General Form of the Equation of the Hyperbola*

The equation $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ represents a hyperbola with transverse axis $y-k=0$ and centre (h,k). Similarly $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ is a hyperbola with centre (h,k) and transverse axis $y-h=0$.

In general, the equation

$$ax^2+by^2+2gx+2fy+c=0$$

represents a hyperbola if $a\neq 0$, $b\neq 0$ and a, b have different signs.

Example. Find the coordinates of the centre and of the foci, the semi-transverse axis, the eccentricity and the equations of the asymptotes of the hyperbola

$$9x^2-4y^2+36x+8y-4=0.$$

Solution. $9x^2-4y^2+36x+8y-4=0$

$$\Rightarrow\quad 9(x^2+4x)-4(y^2-2y)=4$$

$$\Rightarrow\quad 9(x^2+4x+4)-4(y^2-2y+1)=4+36-4$$

$$\Rightarrow\quad 9(x+2)^2-4(y-1)^2=36$$

$$\Rightarrow\quad \frac{(x+2)^2}{4}-\frac{(y-1)^2}{9}=1.$$

Thus $a=2$, $b=3$, $c=\sqrt{13}$, $e=\dfrac{\sqrt{13}}{2}$.

The slopes of the asymptotes are $\pm\frac{3}{2}$. Hence equations of the symptotes are

$$(y-1)=\pm\tfrac{3}{2}(x+2)$$

$$\text{or } 3x-2y+8=0 \text{ and } 3x+2y+4=0.$$

The centre is $(-2,1)$ and the foci are $(\pm\sqrt{13},\ 1)$.

Exercise 8.11.

Find the coordinates of the centre and of the foci, the semi-transverse axis, the eccentricity, and the equation of the asymptotes of the following hyperbolas. Draw their graphs also.

1. $4x^2-9y^2-32x+36y-8=0$
2. $9x^2-16y^2+18x-64y-199=0$
3. $4x^2-y^2+24x+8y+16=0$
4. $16x^2-25y^2-64x-336=0$
5. $16x^2-36y^2+48x+180y-225=0.$

ANSWERS

1. C : $(4,2)$; F : $(4 \pm \sqrt{13}, 2)$; $a=3$; $e=\frac{1}{3}\sqrt{13}$
   A : $2x-3y-2=0$ and $2x+3y-14=0$
2. C : $(-1,-2)$; F : $(-6,-2)$ and $(4,-2)$; $a=4, e=\frac{5}{4}$;
   A : $3x-4y-5=0$ and $3x+4y+11=0$
3. C : $(-3,4)$; F : $(-3 \pm \sqrt{5}, 4)$; $a=1, e=\sqrt{5}$;
   A : $2x-y+10=0$, and $2x+y+2=0$
4. C : $(2,0)$; F : $(2 \pm \sqrt{41}, 0)$; $a=5, e=\frac{1}{5}\sqrt{41}$;
   A : $4x-5y-8=0$ and $4x+5y-8=0$
5. C : $\left(-\frac{3}{2}, \frac{5}{2}\right)$; F : $\left(-\frac{3}{2} \pm \frac{1}{2}\sqrt{13}, \frac{5}{2}\right)$ $a=\frac{3}{2}$, $e=\frac{1}{3}\sqrt{13}$
   A : $4x-6y+21=0$ and $4x+6y-9=0$

By using article on geometric interpretation of $\frac{dy}{dx}$ it can be proved that the equation of the tangent at the point (x_1, y_1) to

(*i*) the circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$,

(*ii*) the parabola $y^2=4ax$ is $yy_1=2a\,(x+x_1)$,

(*iii*) the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$, and

(*iv*) the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

8.18. Find the condition that the line $y=mx+c$ should touch

(*i*) the circle $x^2+y^2=a^2$,

(*ii*) the parabola $y^2=4ax$,

(*iii*) the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and

(*iv*) the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Solution. (*i*) Solving simultaneously the equations $y=mx+c$ and $x^2+y^2=a^2$ we get

$$x^2+(mx+c)^2-a^2=0,$$

$$\text{or}\quad x^2(1+m^2)+2mcx+(c^2-a^2)=0.$$

If the line touches the circle, the roots of this quadratic equation must be equal, which implies that the descriminant

$$(2mc)^2-4(1+m^2)(c^2-a^2)=0$$

$$\text{or}\quad 4m^2c^2=4(c^2-a^2+c^2m^2-a^2m^2)$$

or $c^2=a^2(1-m^2)$ is the required condition.

(*ii*) Solving $y=mx+c$ and $y^2=4ax$ simultaneously we get $(mx+c)^2-4ax=0,$

$$\text{or}\quad m^2x^2+2(mc-2a)x+c^2=0.$$

This equation will have equal roots, if

$$(mc-2a)^2-m^2c^2=0$$

$$\text{or}\quad m^2c^2-4amc+4a^2-m^2c^2=0$$

or $mc=a$ is the required condition.

Parts (*iii*) and (*iv*) can be tackled similarly and the conditions in these cases will be :

$$c^2=b^2+a^2m^2$$

and $c^2=-b^2+a^2m^2$ respectively.

Miscellaneous Exercise 4

1. Find the equation of the circle having A (6, 4) and B $(-2, 7)$ as the end points of one of its diameter.
2. Find the equation of the circle which touches x-axis and is concentric with $2x^2+2y^2-11x+6y-8=0$.
3. Prove analytically that an angle inscribed in a semi-circle is a right angle.

4. Show that the locus of points whose distance from a fixed point is k times $(k \neq 1)$ the distance from another fixed point is a circle.

5. Find the equation of the tongents to the circle $x^2+y^2=10$ at the points whose abscissa is 1.

6. Show that the line $x-y=2$ touches the circle $x^2+y^2=2$ and find the point of contact.

7. Prove that the two circles $x^2+y^2+2ax+c=0$ and $x^2+y^2+2by+c=0$ touch each other if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$.

8. Find the value of r such that the line $x=2y+4$ should be a tangent to the circle $x^2+y^2=r^2$.

9. Find the condition that the line

$$\frac{x}{p}+\frac{y}{q}=1$$

should be a tangent to

(*i*) the circle $x^2+y^2=a^2$,

(*ii*) the parabola $y^2=4ax$,

(*iii*) the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

(*iv*) the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

10. Find the equations of the tangents at the ends of the latus rectum of the parabola $y^2=4ax$.

11. Prove that the parabolas $y^2=4ax$ and $x^2=4by$ intersect at an angle

$$\operatorname{Tan}^{-1}\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}}+b^{\frac{2}{3}})}.$$

12. **Find the equations of the tangent and normal at the point $(at^2, 2at)$ to the parabola $y^2=4ax$.**

13. Show that the equation of the parabola having $(a \cos \varnothing, a \sin \varnothing)$ for its focus and $x \cos \varnothing + y \sin \varnothing + a = 0$ for its directrix is $(x \sin \varnothing - y \cos \varnothing)^2 = 4a (x \cos \varnothing + y \sin \varnothing)$.

14. Show that the line $2x - y + 1 = 0$ touches the parabola $y^2 = 8x$. Find also the point of contact.

15. Find the condition that the line $x \cos \varnothing + y \sin \varnothing = p$ may touch the parabola

$$y^2 = 4ax.$$

16. Show that the line $x \cos \alpha + y \sin \alpha = p$ will touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

if $p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$. Also show that if the condition is satisfied then the point of contact is

$$\left(\frac{a^2 \cos \alpha}{p}, \frac{b^2 \sin \alpha}{p}\right).$$

17. Prove that the equations of the tangent and the normal at the point $(a \cos \theta, b \sin \theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$ and $ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$ respectively.

18. Show that the product of the distances of any point on a hyperbola from its asymptotes is constant.

19. Show that the tangents at the ends of the latera recta of a hyperbola have slopes $\pm e$.

20. Show that the curves

$$\frac{x^2}{36} + \frac{y^2}{20} = 1$$

and $$\frac{x^2}{6} - \frac{y^2}{10} = 1$$

intersect at right angles.

ANSWERS

1. $x^2+y^2-4x-11y+16=0$

2. $\left(x-\frac{11}{4}\right)^2+\left(y+\frac{3}{2}\right)^2=\frac{9}{4}$

5. $x\pm 3y=10$

6. $(1, -1)$

8. $\pm\frac{4}{\sqrt{5}}$

9. (i) $p^2q^2=(p^2+q^2)a^2$

   (ii) $a+\frac{q^2}{p}=0$

   (iii) $p^2q^2=b^2p^2+a^2q^2$

   (iv) $p^2q^2=a^2q^2-b^2p^2$

10. $x\pm y+a=0$

12. $ty=x+at^2$, $y=-tx+2at+at^3$

14. $\left(\frac{1}{2}, 2\right)$

15. $a \operatorname{Sin}^2 \varnothing=-p \operatorname{Cos} \varnothing$.

CHAPTER IX

VECTORS

In **§ 2.2** we introduced the concept of cartesian coordinate system for 2-space. In the following section we are introducing the rectangular cartesian coordinate system for 3-space.

9.1. *Cartesian Coordinate System for 3-Space*

For a rectangular cartesian coordinate system in 3-space we follow the convention and procedure of cartesian 2-space system. Thus to locate points we use ordered triples of real numbers (x, y, z) in which x, y, z are orthogonal projections of OP on the x—axis, the y—axis and the z-axis respectively where O is the origin (figure 9.1).

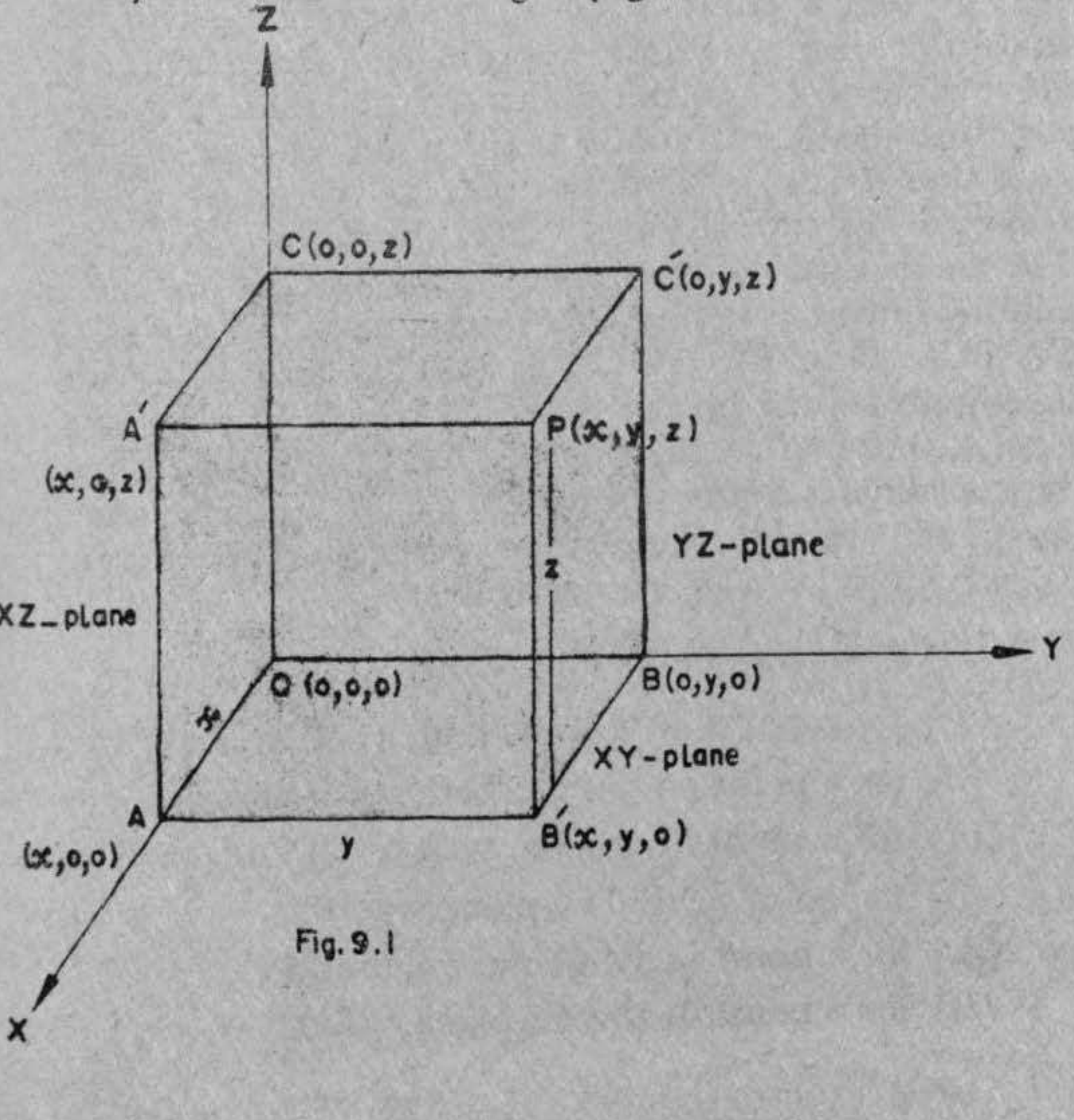

Z
C(o,o,z)
C′(o,y,z)
A′
(x,o,z)
P(x,y,z)
YZ-plane
z
XZ_plane
O(o,o,o)
Y
B(o,y,o)
x
XY-plane
A
(x,o,o)
y
B′(x,y,o)
X

Fig. 9.1

In the above figure the three axes viz. OX, OY and OZ may be imagined to be the edges at a corner of a room or the edges of a brick. OX and OY correspond to the edges formed by the floor and two walls, while OZ corresponds to the vertical edge formed by the two walls. Thus we have three mutually perpendicular straight lines OX, OY and OZ such that they all meet at O (the origin) and $OX \perp OY$, $OY \perp OZ$, $OZ \perp OX$. The standard practice is to follow the *right hand* convention. This means that if x-axis and y-axis are the horizontal and vertical lines in the plane of the paper like the cartesian two space system, then the positive direction of z-axis is the one that points at the reader. In the above figure the positive directions are indicated by arrow heads. The three planes XY, YZ and ZX are also mutually perpendicular.

To plot a point $P(x, y, z)$; (see figure 9.1) we start from the origin, whose coordinates are (0, 0, 0). We move along x-axis to the point A such that $OA=x$; then we move parallel to y-axis to the point B′ in the *xy*-plane such that $AB'=y$. Finally from B′ we move parallel to *z*-axis a distance $B'P=z$. P is the desired point (x, y, z). The shaded part of the above figure is called a rectangular parallelopiped. It has six rectangular faces viz. OAB′B, CA′PC′, B′BC′P, AOCA′, OBC′C and AB′PA′. The coordinates of the points A, B, C (which are on the axes) are $(x, 0, 0)$, $(0, y, 0)$ and $(0, 0, z)$ respectively, whereas the coordinates of the points B′, C′, A′ (which are in the (*xy*-plane, *yz*-plane, *zx*-plane respectively) are $(x, y, 0)$ $(0, y, z)$ and $(x, 0, z)$ respectively.

The students should note that :

(*i*) for a point on x-axis, both $y=0$ and $z=0$,

(*ii*) for a point on y-axis, both $x=0$ and $z=0$,

(*iii*) for a point on z-axis, both $x=0$ and $y=0$,

(*iv*) for point in the *xy*-plane, $z=0$,

(*v*) for a point in *yz*-plane, $x=0$ and

(*vi*) for a point in the *xz*-plane, $y=0$.

9.2. *Distance and Displacement*

Suppose that A and B are any two points on the real line. We consider two concepts associated with the pair A, B.

(1) *Distance between A and B* : denoted by AB associates with every pair of points A and B, a real number which is the measure of the distance between the points A, B only and disregards completely the direction.

(2) *Displacement from A to B* ; denoted by $\overrightarrow{AB}$ associates with every pair of points A and B, which expresses positional distance for which direction is also very important. Displacement is another name for directed distance already explained in § 7.4

9.3. *Scalar and Vector Quantities*

In daily life we usually come across quantities that have magnitude only. They are known as *scalar quantities* or simply *scalars*. For example volume, time, mass, distance, energy and potential are scalars. On the other hand if a quantity needs both magnitude and direction for its specification, it is said to be a *vector quantity*, or simply a *vector*. Displacement, force, velocity, acceleration and momentum are examples of vectors

Notation. In print vectors will be denoted by bold-faced small letters and scalars by ordinary small letters of the alphabet. For example **a**, **b**, **c**, **d**.........are vectors and a, b, c, d.........are scalars.

When writing by hand vectors are usually indicated by underlining the scalars. For instance $\underline{a}, \underline{b}, \underline{c}, \underline{d}$,.........are vectors. Geometrically a vector is indicated by a directed line segment.

9.4. *Addition of Vectors*

Let **a**, **b** be two given vectors represented by the directed line segments $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

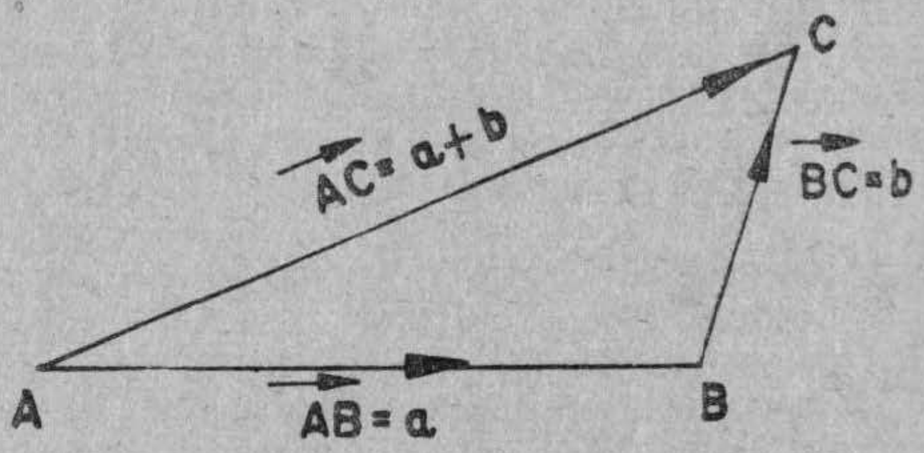

C
$\overrightarrow{AC} = a+b$
$\overrightarrow{BC} = b$
A
$\overrightarrow{AB} = a$
B

Fig. 9.2

The sum of these vectors is defined as the yector $\overrightarrow{AC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{a}+\mathbf{b}.$$

Thus if two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are represented by the (indirection and magnitude)sides AB and BC of **a** △ ABC, then their resultant $\overrightarrow{AC}$ is given by the side AC of the triangle B. This is called the *triangle law of vectors.*

9.5. *Position Vector*

When a vector $\overrightarrow{OP}$ is used to specify the position of a point P relative to another point O, it is called the *position vector* of P for the origin O.

When we say that the position vectors of the vertices of a triangle ABC are **a**, **b** and **c**, we mean that relative to the origin O,

$\overrightarrow{OA} = \mathbf{a}$,

$\overrightarrow{OB} = \mathbf{b}$,

and $\overrightarrow{OC} = \mathbf{c}$.

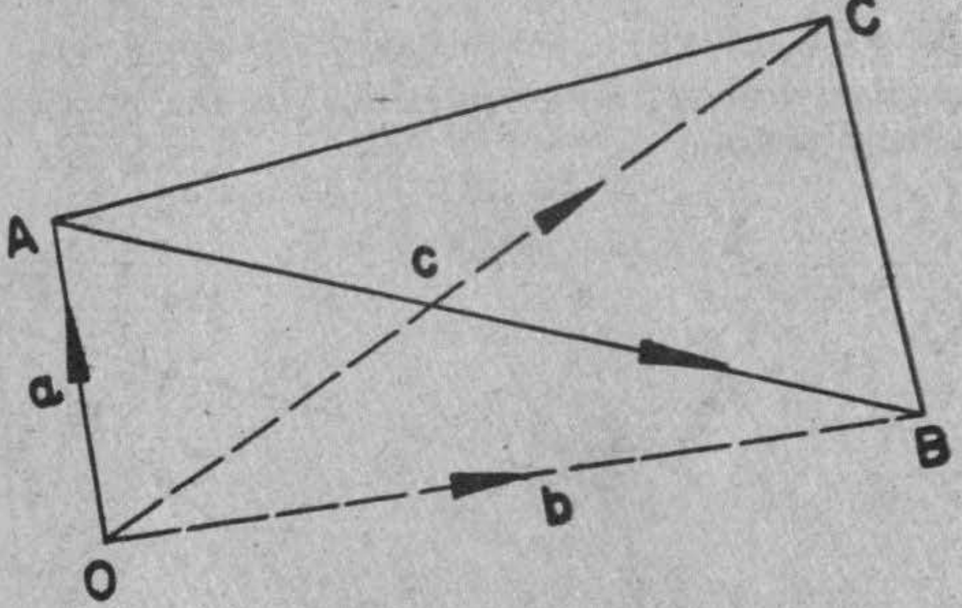


Fig. 9.3

Obviously the position vector $\overrightarrow{AB}$ of B with respect to A is determined as follows

From △ OAB,

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB},$$

$$\text{or} \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \mathbf{b} - \mathbf{a}.$$

9.6. *Equivalence of Vectors and Vector Addition*

Two vectors **a** and **b** are said to be *equivalent* (in symbols we write $\mathbf{a}=\mathbf{b}$) if they are in the same direction and have equal length.

Suppose that A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) and D(x_4, y_4) are points in a plane. The vectocs $\overrightarrow{AB}$ and $\overrightarrow{CD}$ will be equal if and only if the coordinate differences of A and B equal the corresponding coordinate differences of C and D, *i.e.*

$$\overrightarrow{AB}=\overrightarrow{CD} \Leftrightarrow x_2-x_1=x_4-x_3 \text{ and } y_2-y_1=y_4-y_3.$$

Equivalent vectors can be represented by the opposite sides of a parallelogram.

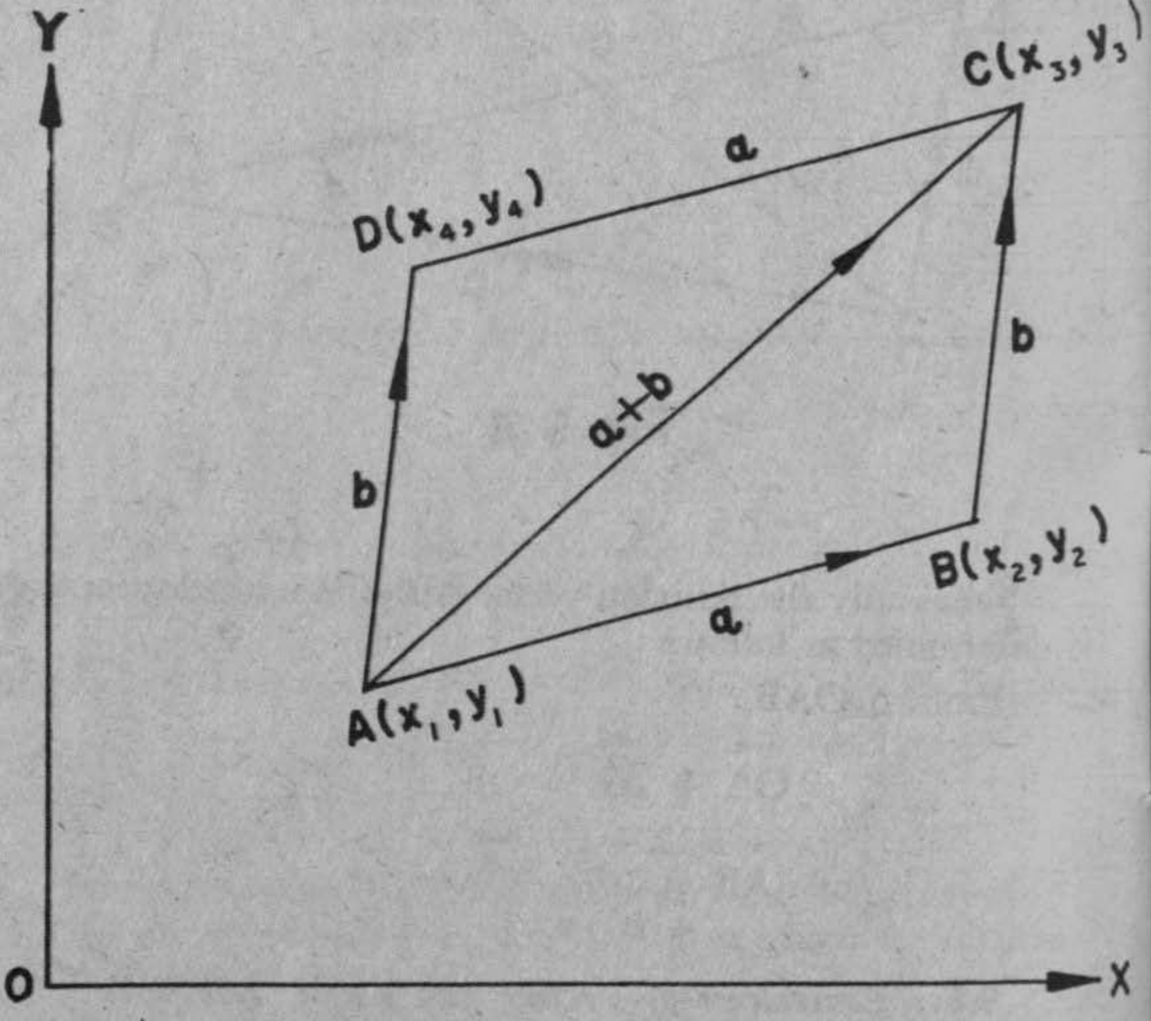


Fig. 9.4

In the above figure $\overrightarrow{AB}=\overrightarrow{CD}$ and $\overrightarrow{AD}=\overrightarrow{BC}$. We also have from this figure

$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC} \text{ (By triangle law)}$$
$$=\overrightarrow{AB}+\overrightarrow{AD} \qquad (\because \overrightarrow{AD}=\overrightarrow{BC}).$$

Hence we conclude that if two vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ can be represented by the adjacent sides of a parallelogram ABCD, then their sum is represented by the diagonal AC of the parallelogram. This is known as the parallelogram law.

From figure 9.4 we also see that if $\overrightarrow{AB}=\mathbf{a}$ and $\overrightarrow{AD}=\mathbf{b}$, then

$$\overrightarrow{AC}=\mathbf{a}+\mathbf{b}.$$

Further, since $\overrightarrow{DC}=\overrightarrow{AB}=\mathbf{a}$ it follows that

$$\mathbf{b}+\mathbf{a}=\overrightarrow{AD}+\overrightarrow{DC}=\overrightarrow{AC}=\mathbf{a}+\mathbf{b} \quad \text{.........(1).}$$

Again, if $\mathbf{c}=\mathrm{CP}$, then we have from the following figure

that
$$\overrightarrow{AP}=(\mathbf{a}+\mathbf{b})+\mathbf{c} \text{ [from } \triangle ACP]$$
$$=\mathbf{c}+(\mathbf{a}+\mathbf{b}) \qquad \text{[by (1)]}.$$

But from $\triangle BCP$,

$$\overrightarrow{BP}=\overrightarrow{BC}+\overrightarrow{CP}=\mathbf{b}+\mathbf{c},$$

so that from $\triangle$ ABP,

$$\overrightarrow{AP}=\overrightarrow{AB}+\overrightarrow{BP}=\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{b}+\mathbf{c})+\mathbf{a}.$$

Thus
$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}) \quad \text{.........(2).}$$

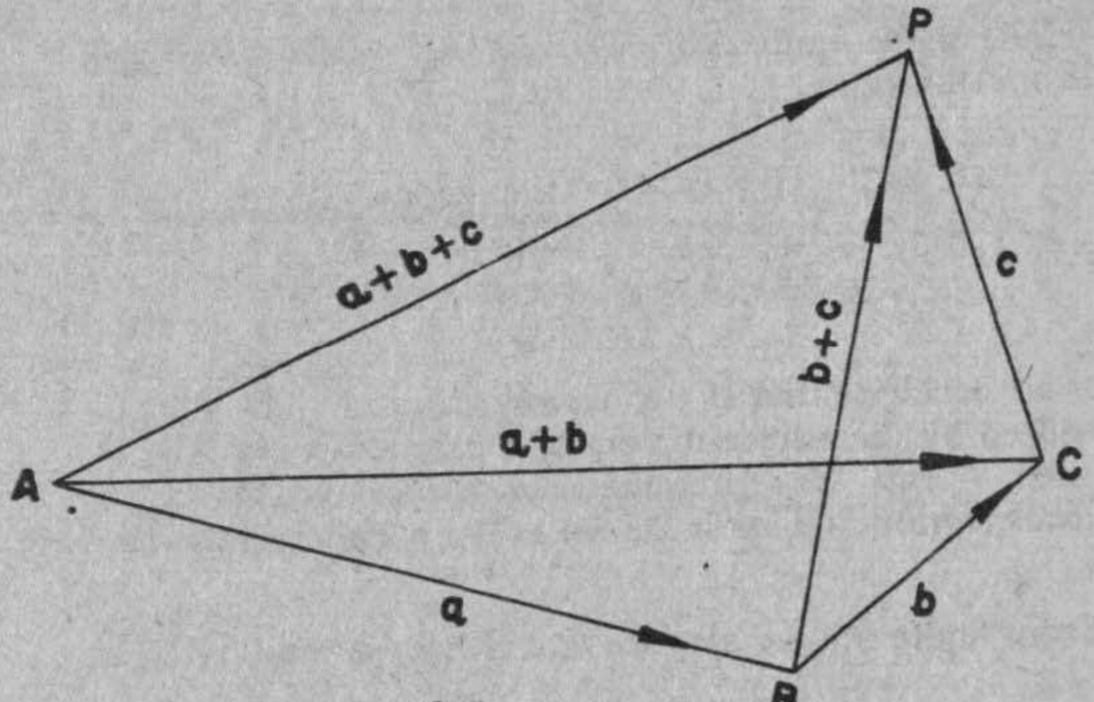


Fig. 9.5

Hence we conclude that the commutative and associative laws hold for the vector addition.

9.7. *Vector Subtraction*

The vector having the same magnitude as **a** but whose direction is opposite to that of **a** is defined as the *negative* of **a**. It is denoted by −**a**.

−**a** is to be understood as the additive inverse of **a**.

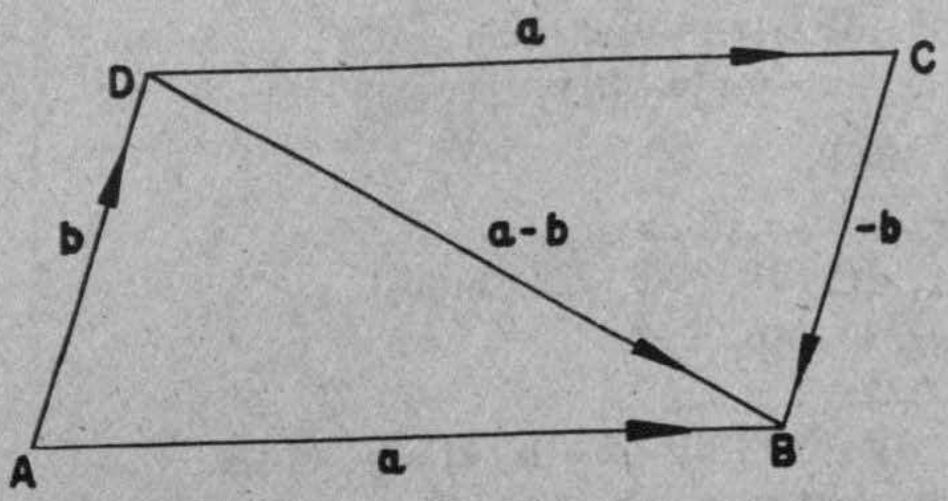


Fig. 9.6

In the above figure $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$ so that

$$\overrightarrow{DC} = \mathbf{a} \text{ and } \overrightarrow{CB} = -\mathbf{b}.$$

Now $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB} = \mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b}$.

Thus to subtract $\mathbf{b}$ from $\mathbf{a}$, we reverse the direction of $\mathbf{b}$ and then add it to $\mathbf{a}$.

Definiton. A *null vector* or a *zero vector* is the one whose magnitude is zero. It is denoted by $\mathbf{0}$. All zero vectors are treated equal irrespective of their directions.

9.8. *Multiplication of a Vector by a Scalar*

If m is a positive real number, m $\mathbf{a}$ *means the vector in the same direction as* $\mathbf{a}$, *but of m times its magnitude.*

$-m\mathbf{a}$ is thus the vector whose magnitude is m times the magnitude of $\mathbf{a}$ and which has direction opposite to that of $\mathbf{a}$.

Definition. A *unit vector* is the one whose magnitude is unity.

$\hat{\mathbf{a}}$ (read as "*a* hat") will denote the unit vector having the same direction as $\mathbf{a}$. Thus we are justified to write

$$\mathbf{a} = a\hat{\mathbf{a}} \text{ and } m\mathbf{a} = m(a\hat{\mathbf{a}}) = ma\hat{\mathbf{a}}.$$

It is also easy to verify that

(*i*) $(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$,

(*ii*) $m(n\mathbf{a}) = mn\mathbf{a} = nm\mathbf{a}$ and

(*iii*) $m(\mathbf{a}+\mathbf{b}) = m\mathbf{a} + m\mathbf{b}$.

Definition. A set of vectors is said to be *coplanar* if a plane can be drawn parallel to all the vectors in the set; otherwise they are called *non-coplanar*.

Example 1. If A (1,2), B (2,4), C (5,6) and D (6,8) are given points, then show that $\overrightarrow{AB} = \overrightarrow{CD}$ but $\overrightarrow{AD} \neq \overrightarrow{BC}$.

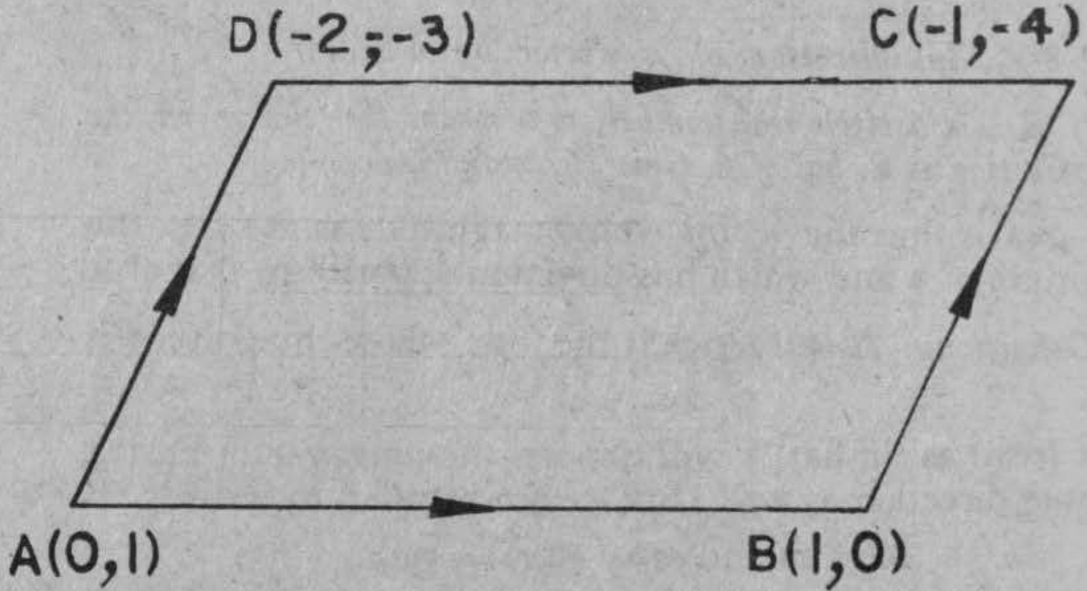
Solution. Now, $2-1=6-5$ and $4-2=8-6$.

Hence $\overrightarrow{AB}=\overrightarrow{CD}$

Again $6-1\neq 5-2$,

hence $\overrightarrow{AD}\neq\overrightarrow{BC}$.

Example 2. Show that the points A $(0,1)$, B $(1,0)$, C $(-1,-4)$ and D $(-2,-3)$ are the vertices of a parallelogram.



Solution. Since $1-0=-1-(-2)$ and

$0-1=-4-(-3)$, it follows that $\overrightarrow{AB}=\overrightarrow{DC}$.

Thus AB = DC and AB // DC, hence ABCD is a parallelogram.

9.9. *Components of a Vector in a Plane*

Any vector r in a plane can be expressed as the sum of two vectors, parallel to any two given vectors.

Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ be unit vectors in the two given directions.

With any point O as origin take $\overrightarrow{OP}=\mathbf{r}$ and on OP as diagonal construct a parallelogram with OA, OB parallel to $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ respectively (Figure 9.7).

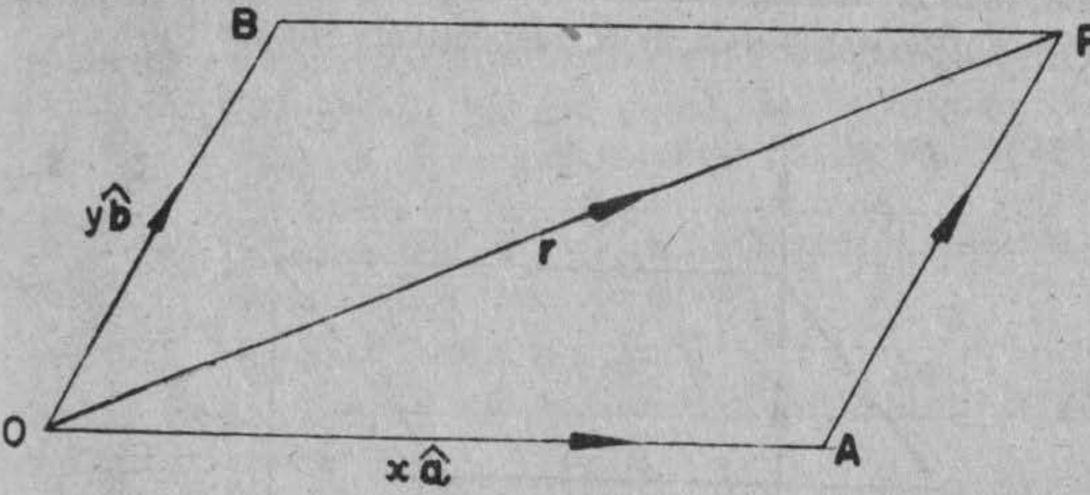


Fig. 9.7

Then if x, y are measures of the sides OA and OB, we have

$$\overrightarrow{OA} = x\hat{\mathbf{a}},\ \overrightarrow{OB} = y\hat{\mathbf{b}}.$$

Now
$$\begin{aligned}\mathbf{r} = \overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \overrightarrow{OA} + \overrightarrow{OB}\\ &= x\hat{\mathbf{a}} + y\hat{\mathbf{b}}\end{aligned}$$

r is called the sum of the vectors $x\mathbf{a}$ and $y\mathbf{b}$. which are called the components of **r** in the given directions $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

This resolution of **r** is unique, since only one parallelogram can be constructed on OP as diagonal with edges parallel to $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

9.10 *Components of a Vector in 3-Space*

Any vector **r** in 3-space can be expressed as the sum of three vectors, parallel to any three given non-coplanar vectors.

Let $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$ be unit vectors in the three given non-coplanar directions. With any point O as origin take $\overrightarrow{OP} = \mathbf{r}$ and on OP as diagonal construct a parallelopiped with OA, OB, OC parallel to $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$ respectively (figure 9.8).

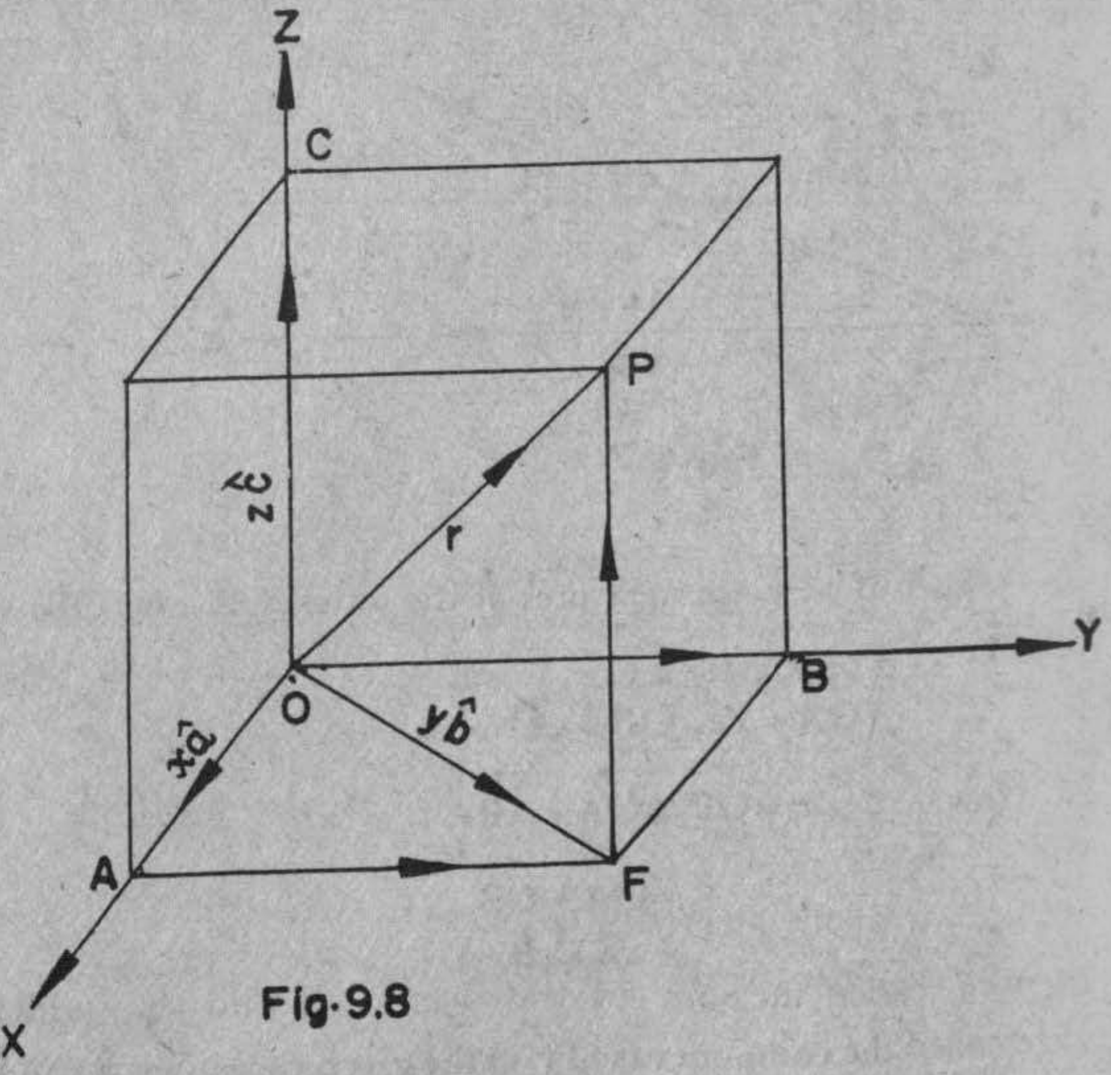


Fig. 9.8

Then if x, y, z are the measures of there edges, we have,

$$\overrightarrow{OA} = x\hat{\mathbf{a}},\ \overrightarrow{OB} = y\hat{\mathbf{b}},\ \overrightarrow{OC} = z\hat{\mathbf{c}}.$$

$$\text{Now } \mathbf{r} = \overrightarrow{OP} = \overrightarrow{OF} + \overrightarrow{FP} = (\overrightarrow{OA} + \overrightarrow{AF}) + \overrightarrow{FP}$$

$$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = x\hat{\mathbf{a}} + y\hat{\mathbf{b}} + z\hat{\mathbf{c}} \qquad (1).$$

$\mathbf{r}$ is said to be the resultant of the three vectors $x\hat{\mathbf{a}}$, $y\hat{\mathbf{b}}$, $z\hat{\mathbf{c}}$, which are called the components of $\mathbf{r}$ in the given directions.

This resolution of r is unique, since only one parallelopiped can be constructed on OP as diagonal with edges parallel to $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$.

Hence we conclude from sections 9.9 and 9.10 that :

(*i*) If two vectors are equal, the components of the one are equal to those of the other, each to each.

(*ii*) Conversely, if two vectors have equal components, they must be equal.

9.11. *The Unit Vectors i, j, k.*

The most important case of resolution of vectors in 3-space is the one in which

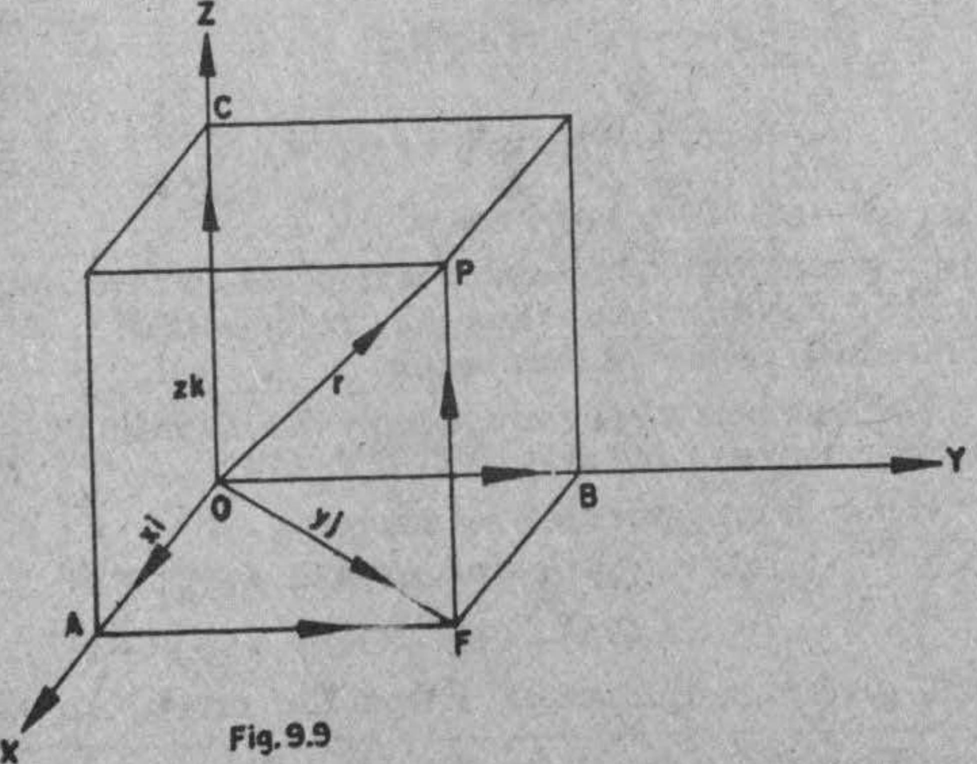


Fig. 9.9

the three given directions are mutually perpendicular. Let i, j, k be unit vectors which are mutually perpendicular and $\mathbf{r}$ is any vector in 3-space.

From figure 9.9 we have

$$\mathbf{r}=\overrightarrow{OP}=\overrightarrow{OF}+\overrightarrow{FP}=(\overrightarrow{OA}+\overrightarrow{AF})+\overrightarrow{FP}$$
$$=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=xi+yj+zk \qquad (2).$$

* It is customary to write $\mathbf{r}=(x,y,z)$ in this case. Thus vector $(2,-4.5)$ means the vector $2i-4j+5k$. If α, β, γ are the angles which $\overrightarrow{OP}$ makes with the x—axis, y—axis and z—axis respectively, $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are said to be the *direction cosines* of OP. (See fig. 9.9). It is also clear from the figure that

$$x=r\cos\alpha,\ y=r\cos\beta,\ z=r\cos v$$

But $\mathbf{r}=r\,\hat{\mathbf{r}}$, so that

$$\hat{\mathbf{r}}=\frac{1}{r}\,\mathbf{r}=\frac{1}{r}\,(xi+yj+zk)$$
$$=\frac{1}{r}(r\cos\alpha i+r\cos\beta j+r\cos v k)$$
$$=\cos\alpha i+\cos\beta j+\cos v k \qquad (3).$$

Thus if a unit vector in 3-space is resolved in three mutually perpendicular directions then the coefficients of i, j, k are the direction cosines of that vector.

The numbers x, y, z are simply the coordinates of the point P relative to the axes OX, OY, OZ.

Hence from figures 9.8 we have

$$r^2=OP^2=OF^2+FP^2=(OA^2+AF^2)+FP^2$$
$$=OA^2+OB^2+OC^2=x^2+y^2+z^2 \qquad (4),$$

which gives the distance of P from the origin O. On the other hand if $\overrightarrow{PQ}$ is any vector in space such that the coordinates of P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then to get the distance between P and Q we proceed as follows.

---

*From this statement it is clear that there is a $1-1$ correspondence between the set of localised vectors $r=(x, y, z)$ and all points P$(x, y\ z)$ in 3 space.

Construct a rectangular parallelopiped on **PQ** as diagonal.

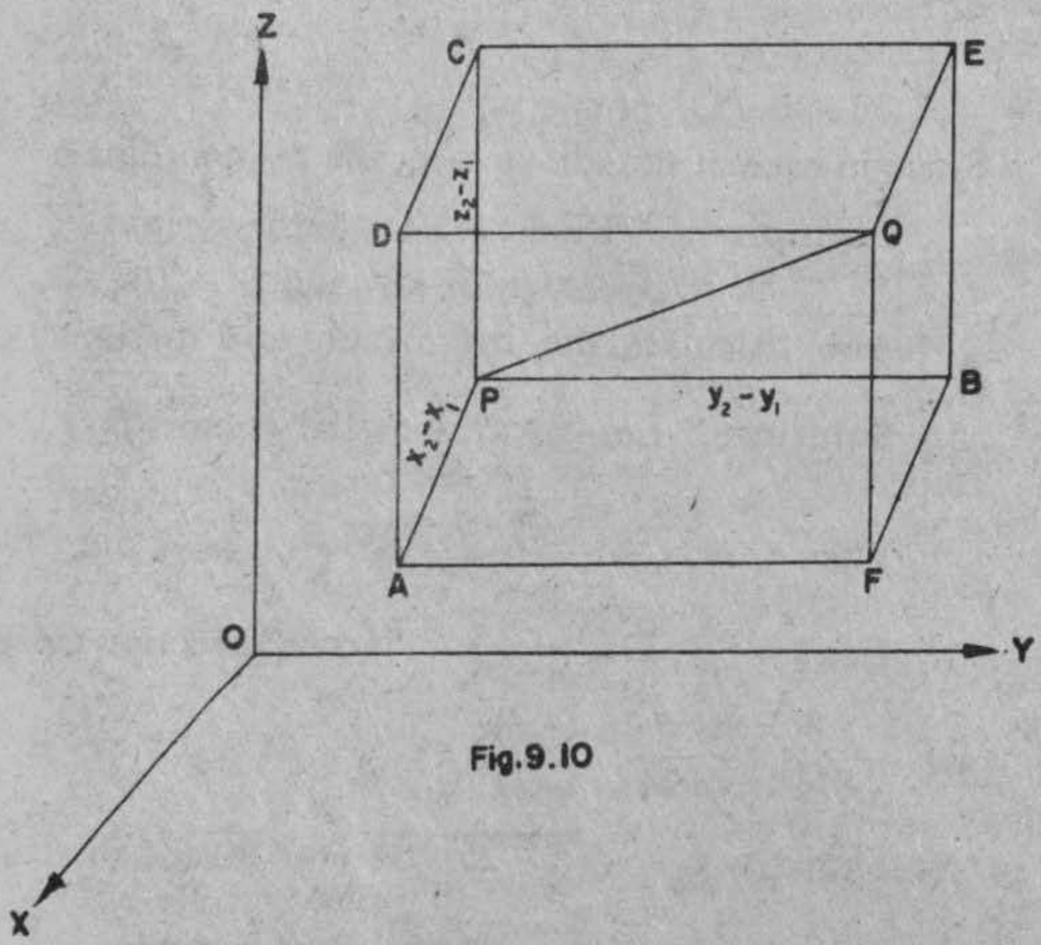


Fig. 9.10

Then from the above figure we have

$$\overrightarrow{PQ}=\overrightarrow{PF}+\overrightarrow{FQ}=(\overrightarrow{PA}+\overrightarrow{AF})+\overrightarrow{FQ}$$

$$= \overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}, \text{ and}$$

$$PQ^2 = PA^2+PB^2+PC^2 = (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2 \ldots\ldots($$

which gives square of the distance between the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2).

If $\mathbf{r}$ is a vector in 2-space (plane) then the results corresponding to results (2) to (5) are as follows:

$$\mathbf{r} = xi + yj \qquad \ldots\ldots(i)$$

$$\mathbf{r} = \text{Cos } \alpha i + \text{Cos}\beta j. \qquad \ldots\ldots(ii)$$

$$r^2 = x^2 + y^2 \qquad \ldots\ldots(iii)$$

$$PQ^2 = (x_2-x_1)^2+(y_2-y_1)^2 \qquad \ldots\ldots(iv).$$

Since in each of the above cases the z—coordinate is zero.

Example 1. Find the sum of the vectors

$$3i+7j-4k,\ i-5j-8k \text{ and } 6i-2j+12k.$$

Also calculate the magnitude and direction cosine of each.

Solution. Let us denote the vectors by $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}$ respectively.

Then $r_1 = \sqrt{3^2+7^2+(-4)^2} = \sqrt{74}$. But Cos $\alpha = \frac{x}{r_1}$

Cos $\beta = \frac{y}{r_1}$, Cos $\nu = \frac{z}{r_1}$. Hence direction cosines of $\mathbf{r}$

are $\frac{3}{\sqrt{74}}, \frac{7}{\sqrt{74}}, \frac{-4}{\sqrt{74}}$.

Similarly $\mathbf{r}_2 = \sqrt{1+25+64} = \sqrt{90}$ and direction cosines

of $\mathbf{r}_2$ are $\frac{1}{\sqrt{90}}, \frac{-5}{\sqrt{90}}, \frac{-8}{\sqrt{90}}$.

$r_3 = \sqrt{36+4+144} = \sqrt{184}$ and direction cosines

of $\mathbf{r}_3$ are $\frac{6}{\sqrt{184}}, \frac{-2}{\sqrt{184}}, \frac{12}{\sqrt{184}}$.

$$\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3 = (3i+7j-4k)+(i-5j-8k)+(6i-2j+\ 2k)$$
$$= 10i+0j+0k = 10i.$$

Since a vector involves both magnitude and direction it is not straightaway possible to say what will be the product of two vectors. In the following sections we are defining two different kinds of products called the *scalar product* and the *vector product*.

9.12. *The Scalar Product*

Definition. The scalar product of two vectors **a** and **b** is defined to be the real number $ab \operatorname{Cos}(\mathbf{a}, \mathbf{b})$, where $\operatorname{Cos}(\mathbf{a}, \mathbf{b})$ is the cosine of the angle between the vectors **a** and **b**. This is written as

$$\mathbf{a} \,.\, \mathbf{b} = ab \operatorname{Cos}(\mathbf{a}, \mathbf{b}).$$

Since $a\,b \operatorname{Cos}(\mathbf{a}, \mathbf{b})$ is a real number the order of the factors may be reversed without changing the product itself *i.e.*

$$\mathbf{b} \,.\, \mathbf{a} = ba \operatorname{Cos}(\mathbf{b}, \mathbf{a}) = ab \operatorname{Cos}(\mathbf{a}, \mathbf{b}); \text{ whence}$$

$$\mathbf{a} \,.\, \mathbf{b} = \mathbf{b} \,.\, \mathbf{a}.$$

From this definition we deduce the following :

(1) If **a** and **b** are mutually perpendicular, then

$$\operatorname{Cos}(\mathbf{a}, \mathbf{b}) = \operatorname{Cos} \frac{\pi}{2} = 0, \text{ so that in this case}$$

$$\mathbf{a} \,.\, \mathbf{b} = 0.$$

(2) If **a** and **b** have the same direction, then

$$\operatorname{Cos}(\mathbf{a}, \mathbf{b}) = \operatorname{Cos} 0 = 1, \text{ thus}$$

$$\mathbf{a} \,.\, \mathbf{b} = ab.$$

(3) If **a** and **b** have opposite directions, then

$$\operatorname{Cos}(\mathbf{a}, \mathbf{b}) = \operatorname{Cos} \pi = -1, \text{ hence}$$

$$\mathbf{a} \,.\, \mathbf{b} = -ab$$

(4) If $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ are unit vectors in the direction of **a**, **b** then $\mathbf{a} \,.\, \mathbf{b} = (1)(1) \operatorname{Cos}(\mathbf{a}, \mathbf{b}) = \operatorname{Cos}(\mathbf{a}, \mathbf{b})$. Thus the scalar product of two unit vectors is simply the cosine of the angle between their directions.

(5) If $\mathbf{a} = \mathbf{b}$. then denoting $\mathbf{a} \;\; \mathbf{a}$ by a^2 we have,

$$\mathbf{a}^2 = \mathbf{a}.. \mathbf{a} = (a)(a) \operatorname{Cos}(\mathbf{a}, \mathbf{a}) = a^2 \operatorname{Cos} 0 = a^2.$$

Hence the square of a vector is equal to the square of it magnitude. In particular

$$i \cdot i = j \,.\, j = k \,.\, k = (i)^2 = (j)^2 = (k)^2 = 1$$

and since i, j, k are mutually perpendicular we get

$$i \cdot j = j \,.\, k = k \cdot i = 0.$$

The scalar product obeys the distributive law, that is

$$\mathbf{a} \,.\, (\mathbf{b}+\mathbf{c}) = \mathbf{a} \,.\, \mathbf{b} + \mathbf{a} \,.\, \mathbf{c}.$$

The proof is beyond the scope of this book.

If $\mathbf{a} = x_1\, i + y_1\, j + z_1\, k$ and $\mathbf{b} = xi + y_2 j + z_2 k$, then from the above results we have

$$(x_1 i + y_1 j + z_1 k) \,.\, (x_2 i + y_2\, j + z_2 k) = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

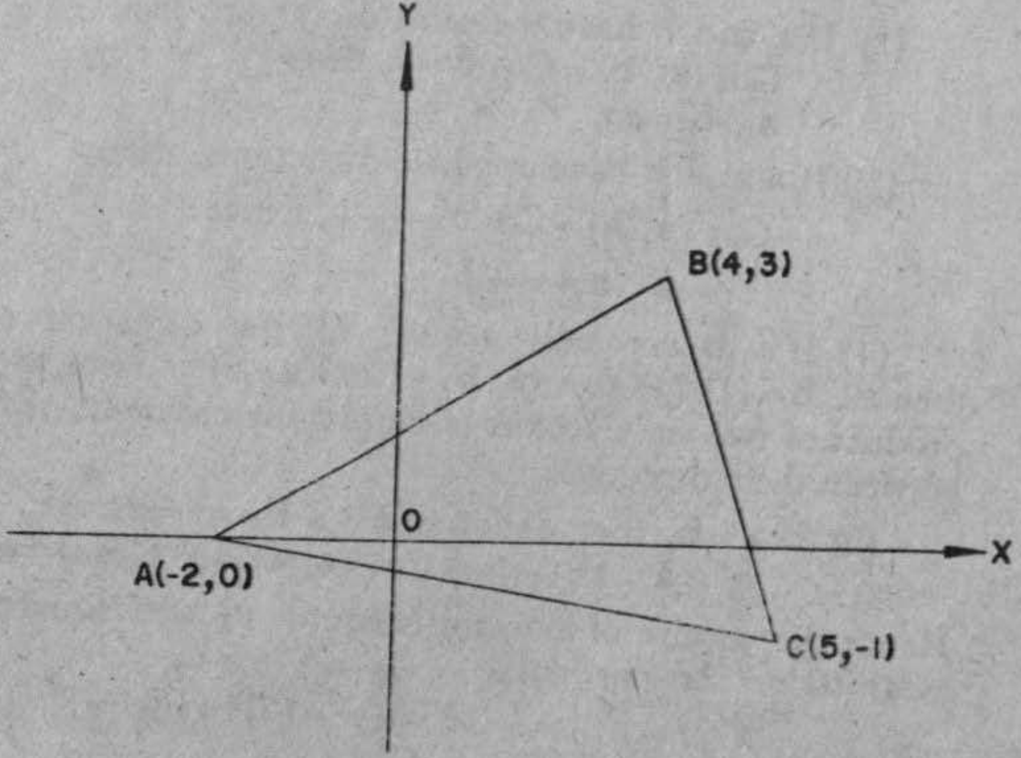
Example 1. Find the modulus or magnitude of the vector $\mathbf{a} = (3, 7)$.

Solution. Now $\mathbf{a} = 3i + 7j$, so that

$$\begin{aligned} a^2 = \mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} &= (3i+7j) \cdot (3i+7j) \\ &= 9 + 49 \quad (\because i.j = 0, j.i = 0) \\ &= 58. \end{aligned}$$

Hence the required length $= \sqrt{58}.$

Example 2. Find Cos A in the △ whose vertices are A $(-2, 0)$, B $(4, 3)$, C $(5, -1)$.



Solution. From $\triangle$ ABC, we have

$$\overrightarrow{AB} = [4-(-2)]i+(3-0)j$$
$$= 6i+3j.$$

$$\overrightarrow{AC} = 7i-1j.$$

Thus $AB^2 = (\overrightarrow{AB})^2 = \overrightarrow{AB} \cdot \overrightarrow{AB} = (6i+3j) \cdot (6i+3j)$

$= 36+9=45$, so that $AB = \sqrt{45}$.

Similarly $AC=\sqrt{50}$.

Now $\overrightarrow{AB} \cdot \overrightarrow{AC} = (AB)(AC) \text{ Cos } A$

$$\Rightarrow \text{Cos } A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{(AB)(AC)} = \frac{(6i+3j)}{\sqrt{45}} \cdot \frac{(7i-1j)}{\sqrt{50}} = \frac{39}{15\sqrt{10}} = \frac{13}{5\sqrt{10}}$$

9.13. *The Vector Product*

Definition. The *vector product* of two vectors **a** and **b** is the vector whose magnitude is the real number ab Sin $(\mathbf{a}, \mathbf{b})$ and which is perpendicular to both **a** and **b**. This is written as

$$\mathbf{a} \times \mathbf{b} = ab \text{ Sin } (\mathbf{a}, \mathbf{b})\, \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane containing **a** and **b**.

If **a** and **b** are in the plane of the paper as shown in figure 9.11 then the positive direction of $\hat{\mathbf{n}}$ is the one which points toawards the reader.

From the above definition it is also obvious that,

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

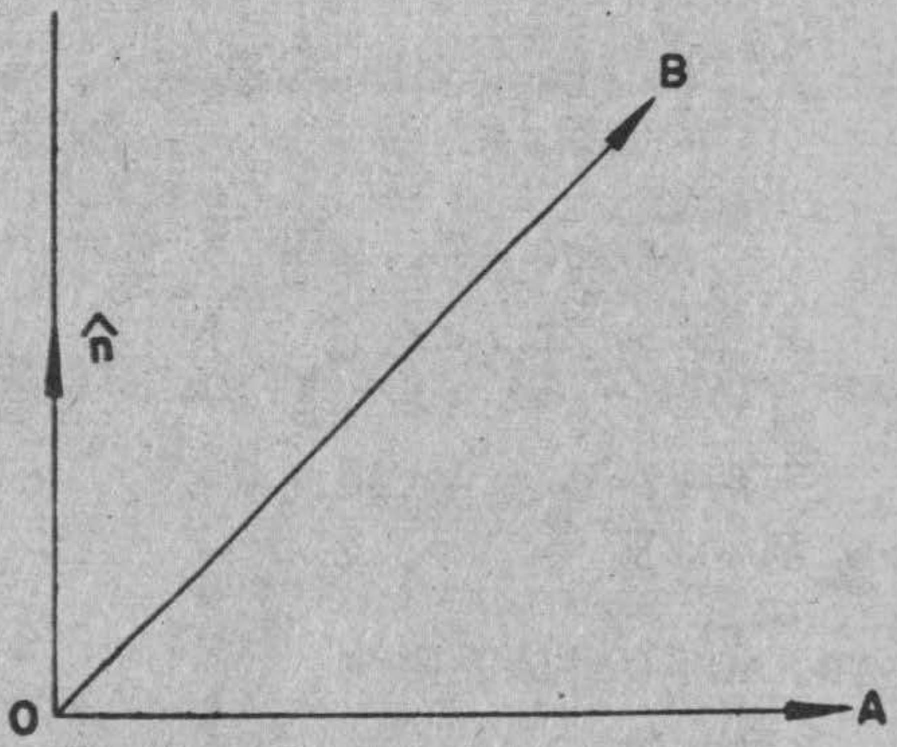


Fig. 9.11

The following results may be deduced from the above definition.

(1). If $\mathbf{a}$ and $\mathbf{b}$ are parallel, Sin $(\mathbf{a}, \mathbf{b})=0$, thus

$$\mathbf{a}\times\mathbf{b}=0$$

is the condition of parallelism of two (finite) vectors. In particular $\mathbf{a}\times\mathbf{a}=0$ for all vectors $\mathbf{a}$.

Hence $i\times i=j\times j=k\times k=0$.

If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular then Sin $(\mathbf{a}, \mathbf{b})=1$, so that in this case $\mathbf{a}\times\mathbf{b}$ is a vector whose magnitude is ab and whose direction is such that the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$ form a right handed system of mutually perpendicular vectors (in the sense of § 9 1).

In particular $i\times j=(1)(1)$ Sin $90°k$ (where k is perpendicular to both i and j)

$$=k.$$

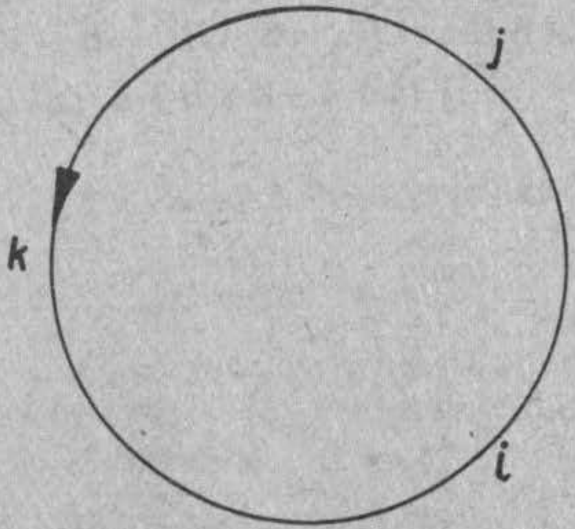
Similarly, $j \times k = i$

and $k \times i = j$,

whereas $j \times i = -k$,

$k \times j = -i$

and $i \times k = -j$.



The students should note that the sign will be positive if the cyclic order of factors is maintained, otherwise it will be negative.

The vector product obeys the distributive law *i.e.*

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

The proof is beyond the scope of this book.

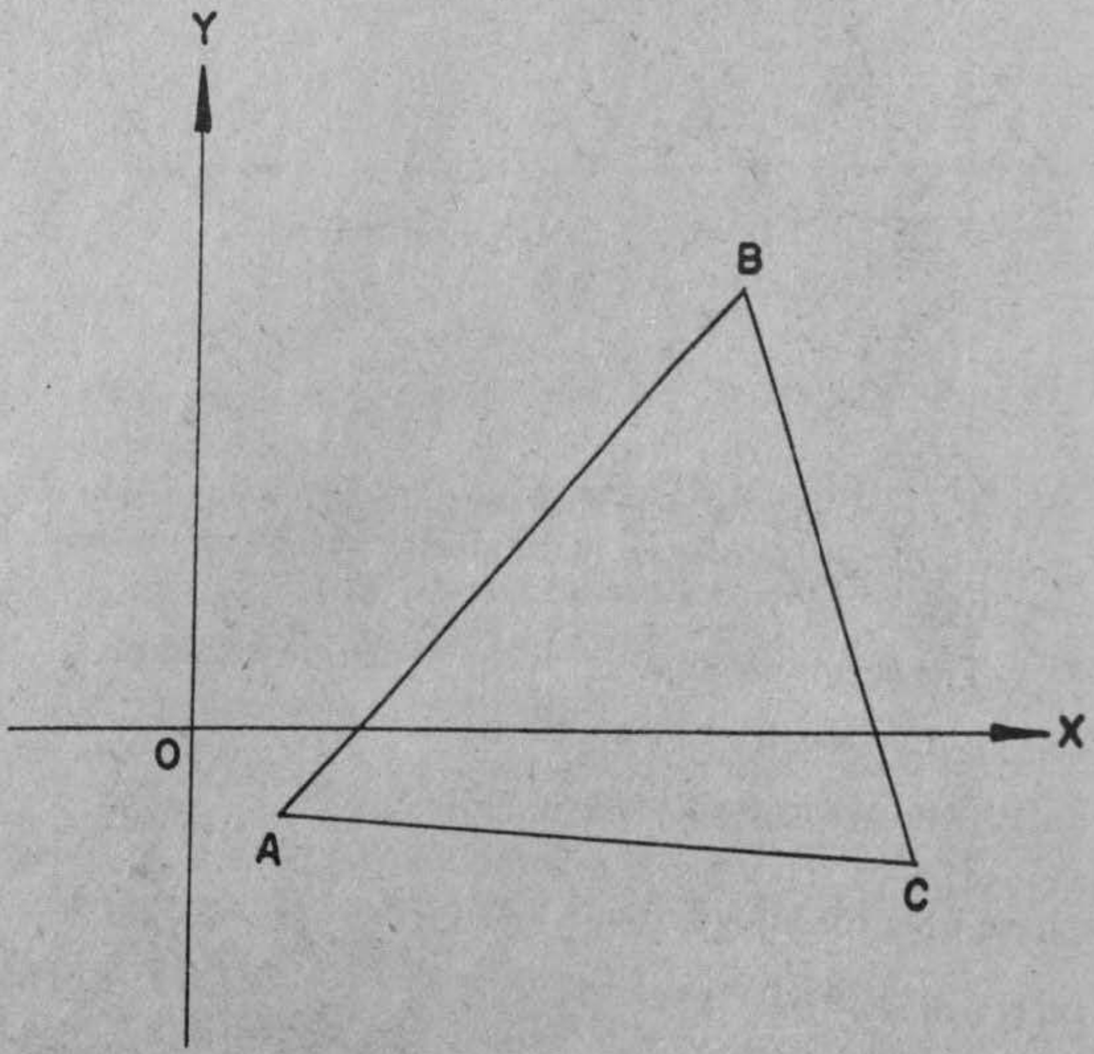
If $\mathbf{a} = x_1\, i + y_1\, j + z_1\, k$ and $\mathbf{b} = x_2\, i + y_2\, j + z_2\, k$, then we have from the above results that

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (x_1\, i + y_1\, j + z_1\, k) \times (x_2\, i + y_2\, j + z_2\, k) \\ &= (y_1 z_2 - z_1 y_2) i + (z_1 x_2 - x_1 z_2) j + (x_1 y_2 - y_1 x_2) k\end{aligned}$$

$$= \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

Example 1. Show that the vector area of the △ ABC, such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, is

$\frac{1}{2}(\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a})$; where O is the origin.



Solution. From the figure we have $\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$,

$\overrightarrow{BC} = \mathbf{c}-\mathbf{b}$ and

$\overrightarrow{AC} = \mathbf{c}-\mathbf{a}$.

We know that $\triangle$ ABC $= \frac{1}{2}$ (AB)(AC) Sin A **n** (**n** is a unit vector parallel to $(\overrightarrow{AC}\times\overrightarrow{AB})$

$$= \tfrac{1}{2}\,(\overrightarrow{AB}\times\overrightarrow{AC})$$
$$= \tfrac{1}{2}\,(\mathbf{b}-\mathbf{a}\times\mathbf{c}-\mathbf{a})$$
$$= \tfrac{1}{2}\,(\mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{a}) \text{ (by disuibutive law)}$$
$$= \tfrac{1}{2}\,(\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a})$$
$$(\because -\mathbf{b}\times\mathbf{a}=\mathbf{a}\times\mathbf{b},\ -\mathbf{a}\times\mathbf{c}=\mathbf{c}\times\mathbf{a} \text{ and } \mathbf{a}\times\mathbf{a}=0).$$

Example 2. Find the unit vector perpendicular to each of the vectors $i+2j+2k$ and $3i-2j-4k$. Also calculate the sine of the angle between these vectors.

Solution. Let $\mathbf{a}=i+2j+2k$ and $\mathbf{b}=3i-2j-4k$,

so that $a = \sqrt{1^2+2^2+2^2}=\sqrt{9}=3$ and

$$b = \sqrt{3^2+2^2+4^2} = \sqrt{29}$$

Now $\mathbf{a}\times\mathbf{b}=(i+2j+2k)\times(3i-2j-4k)$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 3 & -2 & -4 \end{vmatrix} = -4i+10j-8k.$$

Hence the required unit vector $= \dfrac{-4i+10j-8k}{\sqrt{4^2+10^2+8^2}}$

$$= \frac{-4i+10j-8k}{\sqrt{16+100+64}} = \frac{-4i+10j-8k}{\sqrt{180}}.$$

Also $\mathbf{a}\times\mathbf{b}=ab\ \text{Sin}\ (\mathbf{a},\mathbf{b})\ \mathbf{n}$, so that

$$|\,\mathbf{a}\times\mathbf{b}\,|^{*} = ab\ \text{Sin}\ (\mathbf{a},\mathbf{b})$$

$\triangle$ ABC means area of the triangle ABC.

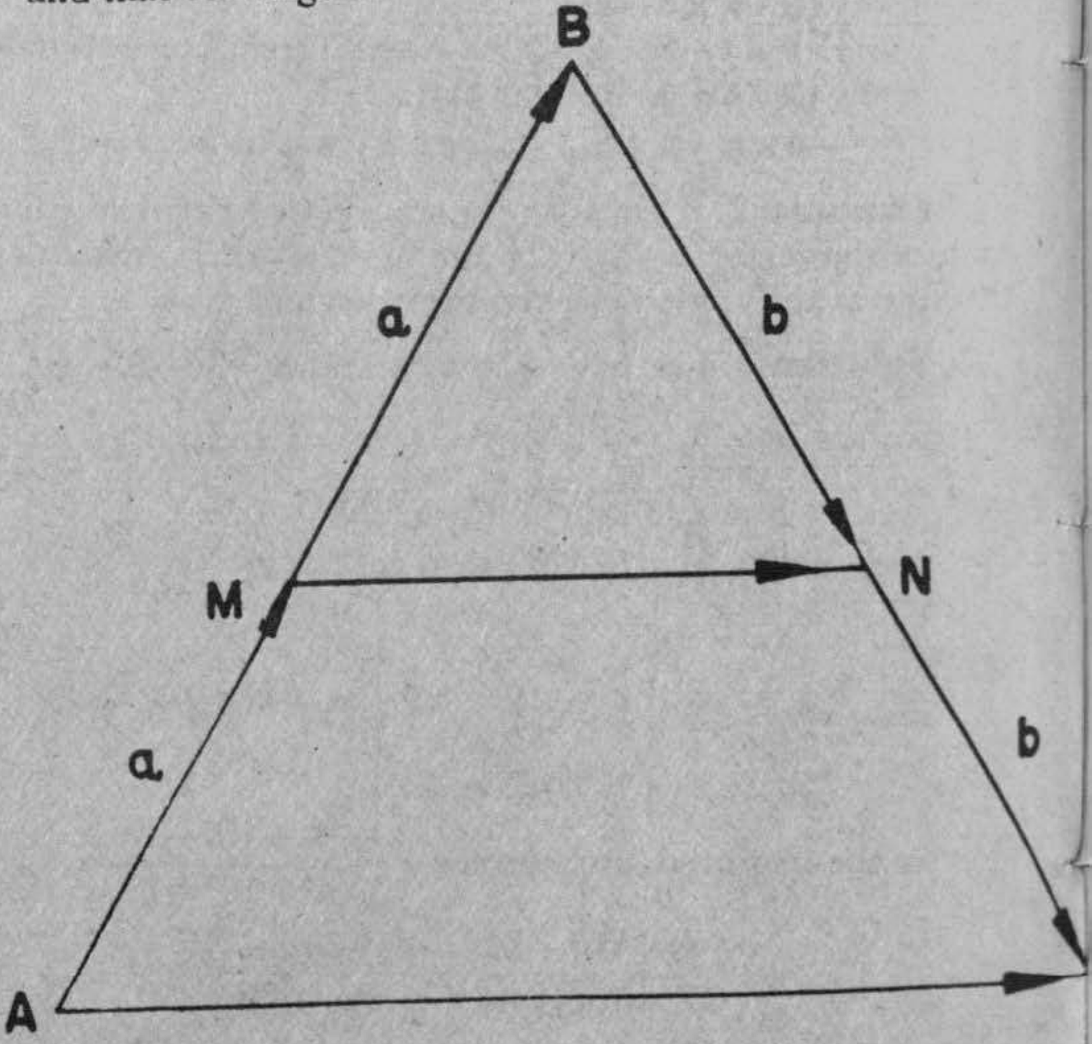
$|\,\mathbf{a}\,|$ = magnitude of $\mathbf{a}=a$.

or $\sqrt{180} = 3\sqrt{29}\ \text{Sin}\ (\mathbf{a}, \mathbf{b})$.

Hence $\text{Sin}\ (\mathbf{a}, \mathbf{b}) = \dfrac{\sqrt{180}}{3\sqrt{29}} = \dfrac{3\sqrt{20}}{3\sqrt{29}} = \sqrt{\dfrac{20}{29}}$.

9.14. The following examples illustrate how the use of vectors simplify the work of analytic geometry.

Example 1. Show that the line joining the mid points of two sides of a triangle is parallel to the third sid and half its length.



Solution. Let M and N be the mid. points of the sid AB and BC of *a* △ ABC. Since $\overrightarrow{AM}$ and $\overrightarrow{MB}$ are equal length and are in the same direction, we have

$$\overrightarrow{AM} = \overrightarrow{MB} = \mathbf{a} \text{ (Say)}$$

Similarly $\overrightarrow{BN} = \overrightarrow{NC} = \mathbf{b}$ (Say)

Thus from $\triangle MBN$,

$$\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = \mathbf{a} + \mathbf{b}.$$

But from $\triangle ABC$,

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 2\mathbf{a} + 2\mathbf{b} = 2(\mathbf{a} + \mathbf{b})$$

Hence $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AC}$ ....(1)

The students should note that equation (1) means two things, *viz.*,

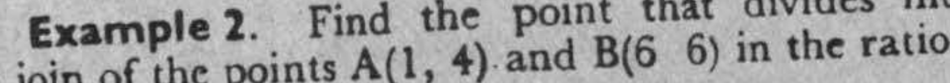
(*i*) length $MN = \frac{1}{2}$ length AC.

(*ii*) $MN \parallel AC$.

Example 2. Find the point that divides internally the join of the points A(1, 4) and B(6 6) in the ratio 1 : 2.

Solution. Let C (x, y) be the required point.

Now $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$, so that



$$(x-1,\ y-4) = \tfrac{1}{3}(6-1,\ 6-4)$$

$$\Rightarrow\ (x-1\ \ y-4) = (\tfrac{5}{3}, \tfrac{2}{3})$$

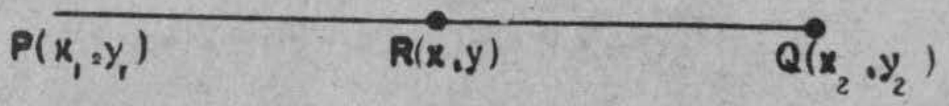
$\Rightarrow \quad x-1=\frac{5}{3},\ y-4=\frac{2}{3}.$

$\Rightarrow \quad x=\frac{8}{3},\ y=\frac{14}{3}.$

Hence the required point is $(\frac{8}{3}, \frac{14}{3})$.

Example 3. Find the mid. point of $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Solution. Let $R\ (x, y)$ be the mid. point of $P\ (x_1, y_1)$ and $Q(x_2, y_2)$.



Now $\overrightarrow{PR}=\frac{1}{2}\ \overrightarrow{PQ} \Rightarrow (x-x_1,\ y-y_1)=\frac{1}{2}\ (x_1-x_u.\ y_2-y_1)$

$\Rightarrow \quad x=\frac{x_1+x_2}{2},\ y=\frac{y_1+y_2}{2}$

Hence the mid. point of the join of P, O is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Exercise 9

1. State which quantity is a scalar and which is a vector.

(*i*) Age of a person

(*ii*) A force

(*iii*) A velocity

(*iv*) Number of a house in a street

(*v*) Distance between Lahore and Islamabad.

2. Draw the following vectors :

(*i*) $\mathbf{2}, \mathbf{3}, \mathbf{5}$

(*ii*) $\mathbf{a}, \mathbf{b}, \mathbf{a}+\mathbf{b}$

(*iii*) $\mathbf{a}, \mathbf{b}, \mathbf{a}-\mathbf{b}$

(*iv*) $\frac{1}{2}\mathbf{a}, 3\mathbf{b}. \frac{1}{3}\mathbf{a}+3\mathbf{b}$

(*v*) $-\mathbf{a}, -2\mathbf{b}, -\mathbf{a}-2\mathbf{b}$.

3. Find the vector $\overrightarrow{AB}$ when

(*i*) A is $(2, -3)$ and B is $(-1, 4)$

(*ii*) A is $(0, -1)$ and B is $(-1, 0)$

(*iii*) A is (x_1, y_1) and B is (x_2, y_2).

4. Find the sum of the following vectors

(*i*) $2i+3j+k, -i+2j+4k, i+j-k$

(*ii*) $i+j+k, 2i+2j+2k, 3i+3j+3k$

(*iii*) $7i-8j, -5i+3j$.

5. Let A, B, C be the points given by $(3, 7)$, $(4, 7)$ and $(2, 7)$ respectively, show that $\overrightarrow{AB}=\overrightarrow{CA}$.

6. If $O=(0, 0)$, $A=(-1, 2)$, $B=(5, -3)$. $C=(4, -1)$ and $D=(6, -5)$. Show that $\overrightarrow{OA}=\overrightarrow{BC}=\overrightarrow{DB}$

7. If $A=(1, 2)$. B $(-1, 5)$. $C=(4, -1)$. Find D and E such that $\overrightarrow{AB}=\overrightarrow{CD}$ and $\overrightarrow{AC}=\overrightarrow{BE}$.

8. Verify each of the following assertions

(*a*) $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{BA} = \overrightarrow{DC}$

(*b*) $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{CD} = \overrightarrow{AB}$

(*c*) $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{CD} = \overrightarrow{EF} \Rightarrow \overrightarrow{AB} = \overrightarrow{EF}$

(*d*) $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AB} = \overrightarrow{CE} \Rightarrow D = E$.

9. Given A (0, 0), B (0, 2), C (8, 0). Find D such that

$$\overrightarrow{AB} = \overrightarrow{CD} \text{ and } \overrightarrow{AC} = \overrightarrow{BD}.$$

10. Vectors drawn from the origin to points S, T, U are respectively $\mathbf{a}$, $\mathbf{b}$, $4\mathbf{a} - 3\mathbf{b}$. Find $\overrightarrow{ST}$ and $\overrightarrow{TU}$, and show that S, T, U are collinear.

11. Find the point that divides the segment PQ,

P (−1, −4) and Q (−5, 6), in the ratio 3 : 4

12. Find the length of the medians of the triangle formed by A (4, 2), B (1, −2), C (−2, 6).

13. Given the quadrilateral formed by A (−2, −3), B (3, −1), C (6, 3), D (−1, 5). Find the mid. points of the sides.

14. Show that the medians of a triangle are concurrent.

15. Show that the line segment joining the mid. points of opposite sides of a quadrilateral bisect each other.

16. Find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$, when

(*i*) $\mathbf{a} = 2i + 3j + 4k$, $\mathbf{b} = i - j + k$

(*ii*) $\mathbf{a} = i + j + k$, $\mathbf{b} = -5i + 2j - 3k$

(*iii*) $\mathbf{a}=4i+2j+k,\quad \mathbf{b}=2i+1j-k$

(*iv*) $\mathbf{a}=-2i-2j-2k \qquad =3i+3j+3k$

(*v*) $\mathbf{a}=-i-j-k,\quad \mathbf{b}\ \ 2i+j.$

17. Find the length of the following vectors.

(*a*) $3i+7j$ (*b*) $3i-7j$

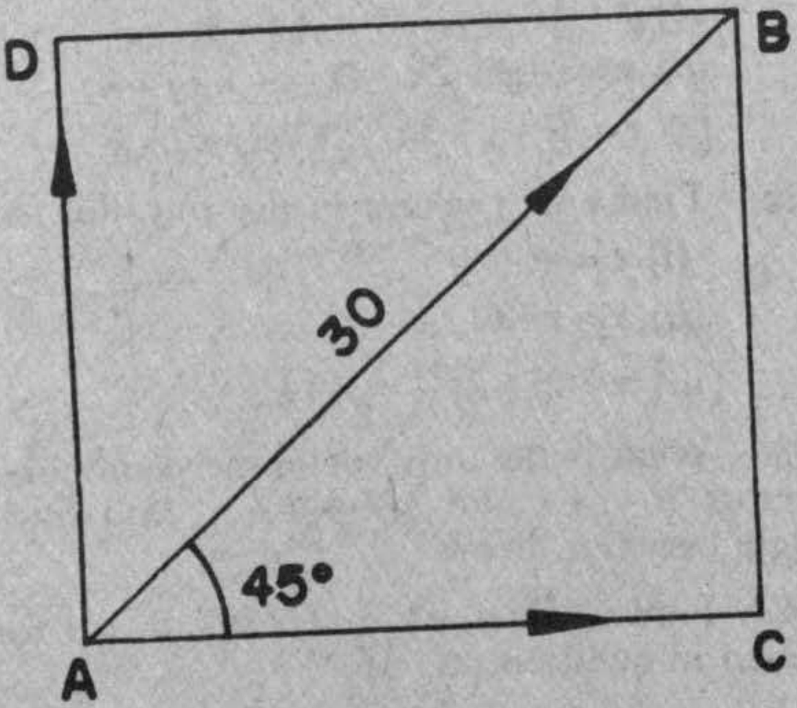
(*c*) $2i+3j+k$ (*d*) $-i-j-k$

(*e*) $-2i-4j+3k.$

18. Find Cos A in ΔABC, when $A=(-2,0), B=(4,3)$, $C=(5,-1)$.

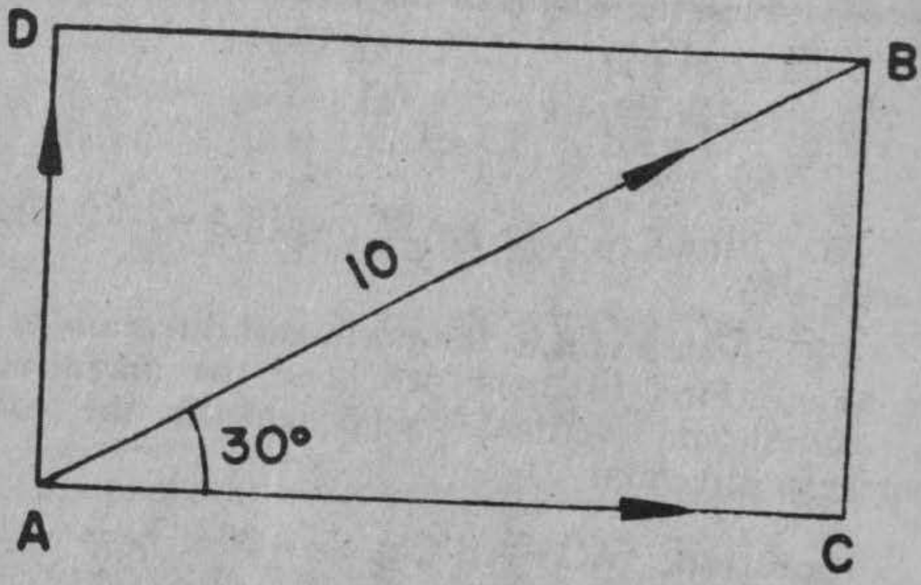
19. A ship sails in the north-east direction at 30 knots an hour. Find in knots per hour the magnitude of the horizontal and vertical components in the eastern and northern directions

Solution. $AC=AB\text{ Cos }45=30\times\frac{1}{\sqrt{2}}=21{\cdot}3$ knots per hour.



$AD = AB \text{ Sin } 45 = 30 \times \frac{1}{\sqrt{2}} = 21.3$ knots per hour.

20. In the adjoining figure find AC and AD.



21. Decide whether **r** and **t** are perpendicular ?

(a) $\mathbf{r} = 2i - 3j, \quad \mathbf{t} = 6i + 5j$

(b) $\mathbf{r} = 6i + 7j \quad \mathbf{t} = -\frac{7}{2}i + 3j$

(c) $\mathbf{r} = \frac{5}{2}i + \frac{2}{3}j, \quad \mathbf{t} = \frac{3}{10}i - \frac{9}{8}j$

(d) $\mathbf{r} = i + j + 2k, \quad \mathbf{t} = -i - j + k$

(e) $\mathbf{r} = 2i + j + 3k, \quad \mathbf{t} = 2i + j - k.$

22. Find a unit vector in the parallel to **r**, when,

(i) $\mathbf{r} = 4i + 3j$

(ii) $\mathbf{r} = i + 6j$

(iii) $\mathbf{r} = 2i + 3j - \sqrt{3k}.$

23. What is the unit vector perpendicular to each of the vectors $2i - j + k$ and $3i + 4j - k$. Also find the sine of the angle between them.

24. Find Cos (**a**, **b**) and Sin (**a**, **b**) for the vector **a**, **b** given in question no. 16.

25. Show that the mid. points of the sides of a quadrilateral are the vertices of a parallelogram.

26. Prove that the diagonals of a parallelogram bisect each other.

27. Prove that the line joining one vertex of a parallelogram to the mid. point of an opposite side is trisected by a diagonal.

28. Find the vector area of the triangles ABC, when

(*i*) $\overrightarrow{OA} = 2 \qquad \overrightarrow{OB} = 3 \qquad \overrightarrow{OC} = 5$

(*ii*) $\overrightarrow{OA} = -4 \qquad \overrightarrow{OB} = 7 \qquad \overrightarrow{OC} = 3$

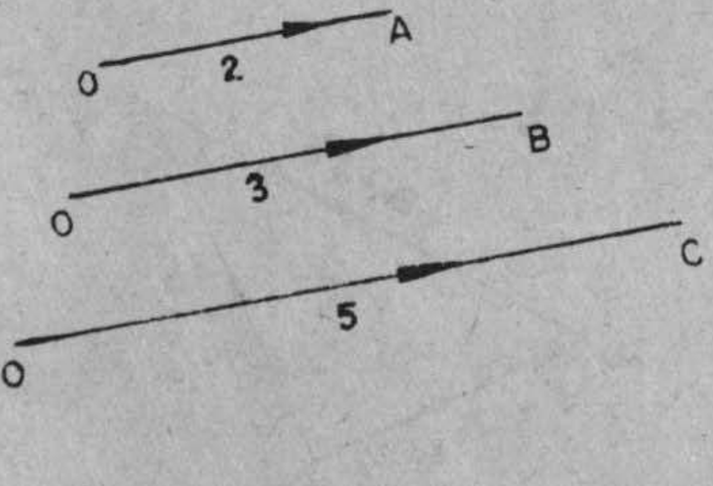
(*iii*) $\overrightarrow{OA} = -2 \qquad \overrightarrow{OB} = -3 \qquad \overrightarrow{OC} = 6.$

where O is the origin of vectors.

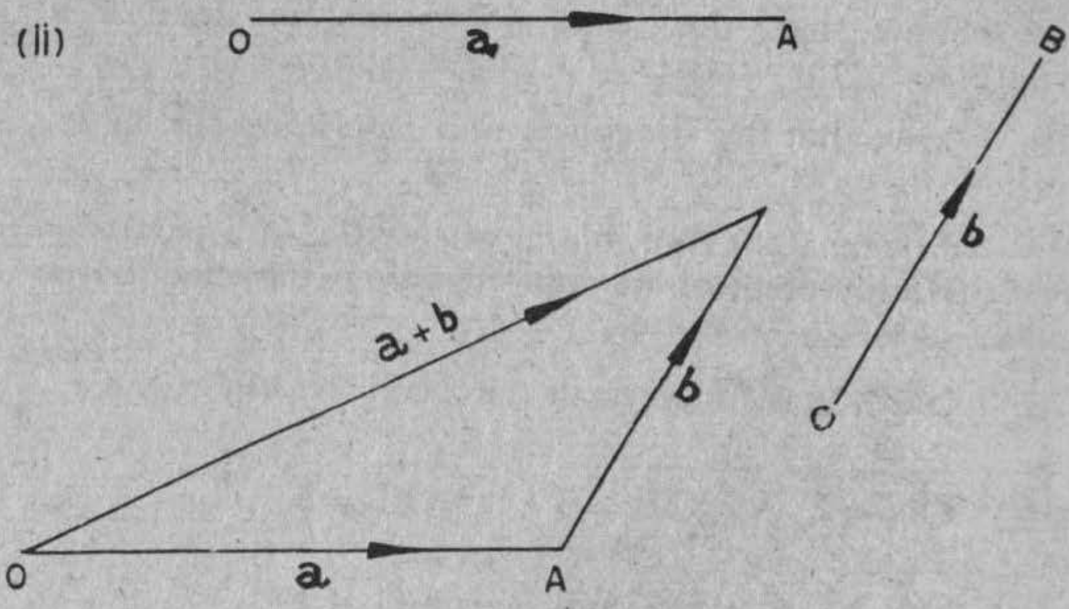
ANSWERS

1. (*i*) Scalar (*ii*) Vector (*iii*) Vector
   (*iv*) Scalar (*v*) Scalar.

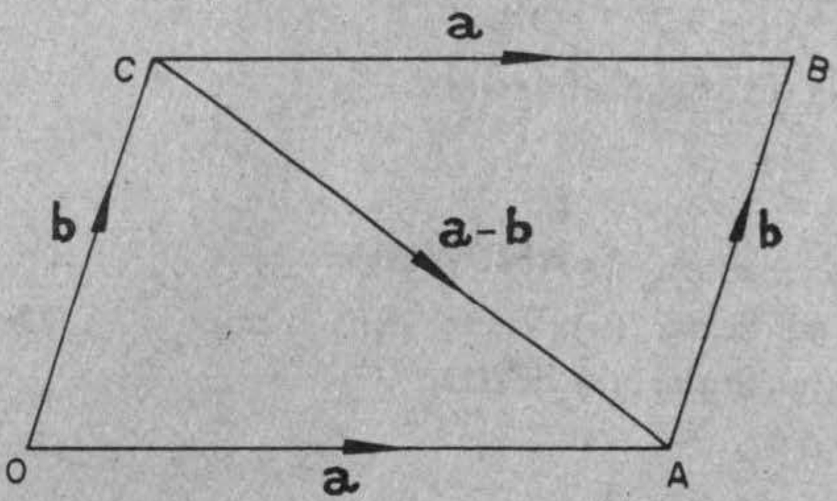
2. (*i*)



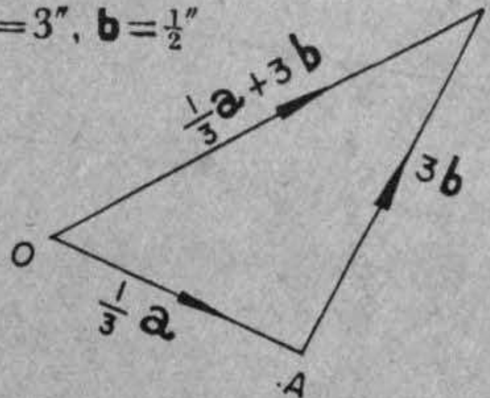
(ii)


O
a
A
B
b
C
a + b
b
O
a
A

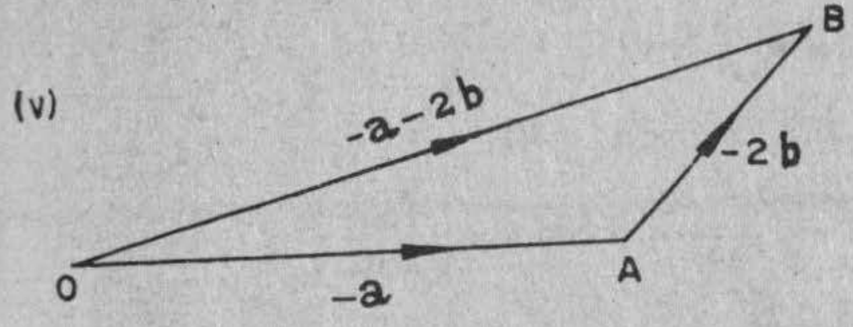
(iii)


a
C
B
b
a-b
b
O
a
A

(*iv*) Take **a** $= 3''$, **b** $= \frac{1}{2}''$


$\frac{1}{3}a + 3b$
$3b$
O
$\frac{1}{3}a$
A

(*v*) Take $\mathbf{a} = 2''$, $\mathbf{b} = \frac{1}{2}''$



3. (*i*) $(-3, 7)$ (*ii*) $(-1, 1)$

(*iii*) $(x_2 - x_1, y_2 - y_1)$

4. (*i*) $2i + 6j + 4k$

(*ii*) $6i + 6j + 6k$

(*iii*) $2i - 5j$.

7. $D = E = (2, 2)$ 9. $(8, 2)$

10. $-\mathbf{a} + \mathbf{b}$, $4(\mathbf{a} - \mathbf{b})$ 11. $(-\frac{19}{7}, \frac{2}{7})$

12. $\frac{9}{2}, 6, \frac{15}{2}$ 13. $(\frac{1}{2}, -2), (\frac{9}{2}, 1), (\frac{5}{2}, 4), (-\frac{3}{2}, 1)$

16. (*i*) 3, $7i + 2j - 5k$ (*ii*) -6, $-5i - 2j + 7k$.

(*iii*) 15, $-6i + 6j + 12k$

(*iv*) -18, null

(*v*) -3, $i - 2j + k$

17. (*a*) $\sqrt{58}$ (*b*) $\sqrt{58}$ (*c*) $\sqrt{14}$

(*d*) $\sqrt{3}$ (*e*) $\sqrt{29}$

18. $\frac{13}{5\sqrt{10}}$ 20. 8.66 and 5

21. (*a*) No (*b*) Yes

(*c*) No (*d*) Yes

(*e*) No

22. (*i*) $\frac{1}{5}(4i+3j)$

(*ii*) $\frac{1}{\sqrt{37}}(i+6j)$

(*iii*) $\frac{1}{4}(2i+3j-\sqrt{3}\,k)$

23. $\frac{-3i+5j+11k}{\sqrt{155}}$, $\text{Sin}\ \theta=\sqrt{\frac{155}{156}}$

24. (*i*) $\sqrt{\frac{3}{29}}, \sqrt{\frac{26}{29}}$

(*ii*) $\frac{-2\sqrt{3}}{\sqrt{38}}; \sqrt{\frac{13}{19}}$

(*iii*) $\frac{5}{7}, \frac{\sqrt{24}}{7}$

(*iv*) $-1\ ;\ 0$

(*v*) $-\sqrt{\frac{3}{5}}; \sqrt{\frac{2}{5}}$

28. (*i*) $\frac{1}{2}(3\times5+2\times3+5\times2)$

(*ii*) $\frac{1}{2}(7\times3+7\times4+4\times3)$

(*iii*) $\frac{1}{2}(6\times3+2\times3+2\times6)$

CHAPTER X

APPLICATION OF CALCULUS TO GEOMETRY

10.1. *Geometrical Interpretation of Derivative*

Let P (x, y) be a point on a curve * whose equation is $y=f(x)$. Also suppose that Q (x_o, y_o) is another point on the curve near P. The line passing through P and Q is called a *secant*.

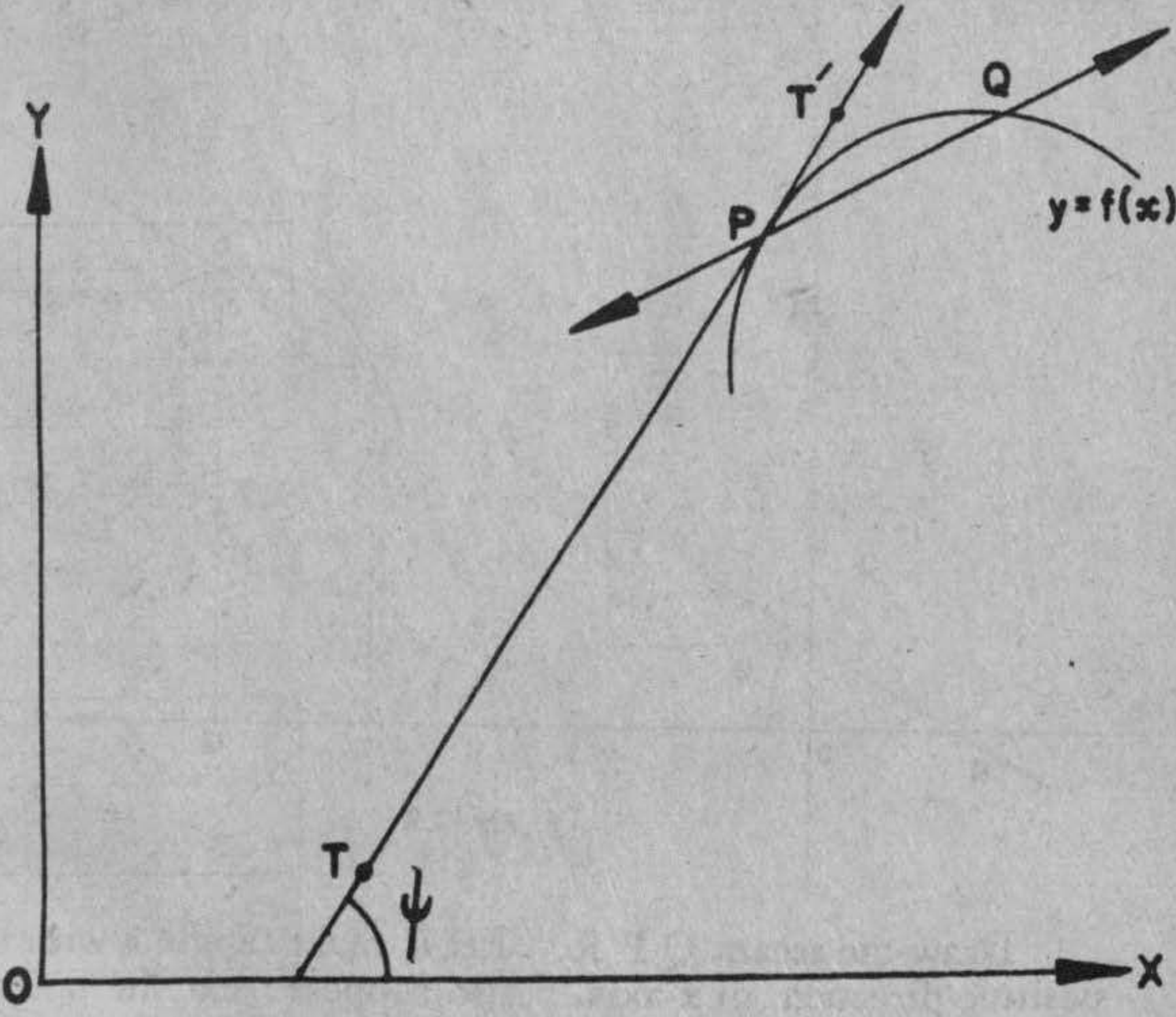


Fig. 10.1

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* In the sequel, for the sake of convenience, we shall sa "the curve $y = f(x)$."

If Q starts moving towards P and keeps moving till it coincides with P, then secant P Q will coincide with the line T P T′ *i.e.* if Q → P, PQ → T P T′. The line T P T′ is said to be the *tangent* at P to the curve $y=f(x)$. *Thus the tangent at a point P to a curve is the limiting position of the secant PQ when Q → P.* In figure 10.1 $\tan\psi$ is the slope of the tangent at P.

10.2. Show that $\frac{dy}{dx}$ at a point of a curve is the slope of the tangent at that point.

Proof. Let P (x,y) and Q $(x+\delta x, y+\delta y)$ be two neighbouring points on a curve whose equation is $y=f(x)$.

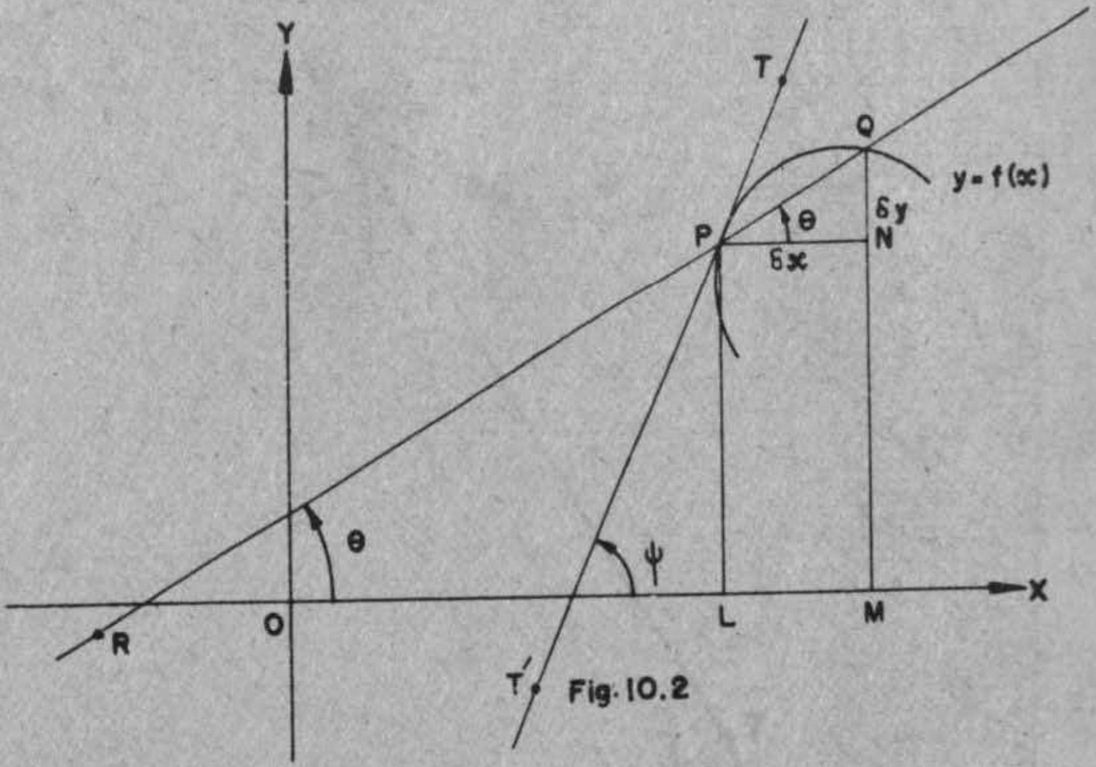


Fig. 10.2

Draw the secant Q P R. Let it make angle θ with the positive direction of x-axis. Also suppose that the tangent T P T′ at P makes angle ψ with the positive direction of x—axis. From P and Q draw PL and QM perpendiculars on x—axis. Also draw PN // OA.

Clearly PNQ is a right triangle such that $\tan\theta=\frac{\delta y}{\delta x}$.

Taking limits when Q → P, so that $\delta x \to 0$ and $\theta \to \delta$, we have

$$\lim_{\theta \to \psi} \tan\theta = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Hence $\tan\psi = \frac{dy}{dx}$.

10.3. *The Equations of Tangent and Normal*

If $f(x)$ is a function defined and differentiable on R then equation of the tangent to the curve $y=f(x)$ at the point (x_1, y_1) on the curve is

$$y-y_1=m(x-x_1)$$

$$\text{where } m=\frac{dy}{dx} \text{ at } x=x_1$$

and the equation of the normal at (x_1, y_1) is

$$y-y_1=-\frac{1}{m}(x-x_1).$$

Example 1. Find equations of the tangent and the normal to the curve given by the equation $y=x^3-2x$ at the point $(1,-1)$ on it.

Solution. Here $y=x^3-2x$.

$$\therefore \frac{dy}{dx}=3x^2-2=3-2=1 \text{ at } x=1$$

Equation of tangent at $(1,-1)$ is

$$y+1=1\,(x-1)$$

$$\text{or} \quad x-y-2=0.$$

Equation of normal at $(1,-1)$ is

$$y+1=-(x-1)$$

$$\text{or} \quad x+y=0.$$

Example 2. Show that the equation of the tangent to the curve $ax^2+by^2+2gx+2fy+c=0$ ......(1)

at (x_1, y_1) is

$$axx_1+byy_1+g(x+x_1)+f(y+y_1)+c=0 \quad \text{......(2)}.$$

Solution. Differentiating (1) w.r.t. x, we have

$$2ax+2byy'+2g+2fy'=0$$

$$\Rightarrow 2(by+f')y'=-2(ax+g)$$

$$\Rightarrow y'=-\frac{ax+g}{by+f}\cdot$$

$$\text{At } (x_1, y_1),\ y' = \frac{ax_1+g}{by_1+f}\cdot$$

Thus equation of the tangent at (x_1, y_1) is

$$y-y_1=-\frac{ax_1+g}{by_1+f}(x-x_1)$$

or $axx_1+byy_1+gx+fy=ax_1^2+by_1^2+gx_1+fy_1$.

Adding gx_1+fy_1+c on both sides, we have

$$\begin{aligned}&axx_1+byy_1+g(x+x_1)+f(y+y_1)+c\\&=ax_1^2+by_1^2+2gx_1+2fy_1+c\\&=0[\text{as } (x_1, y_1) \text{ lies on (1)}].\end{aligned}$$

Thus equation of the tangent to the curve (1) at the point (x_1, y_1) on it is given by (2).

10.4. *The Intersection of Two Curves*

Let $y=f_1(x)$ and $f_2(x)$ be defined and differentiable on R and let c_1 and c_2 represent them respectively. The points of intersection of the two curves can be found by solving their equations simultaneously.

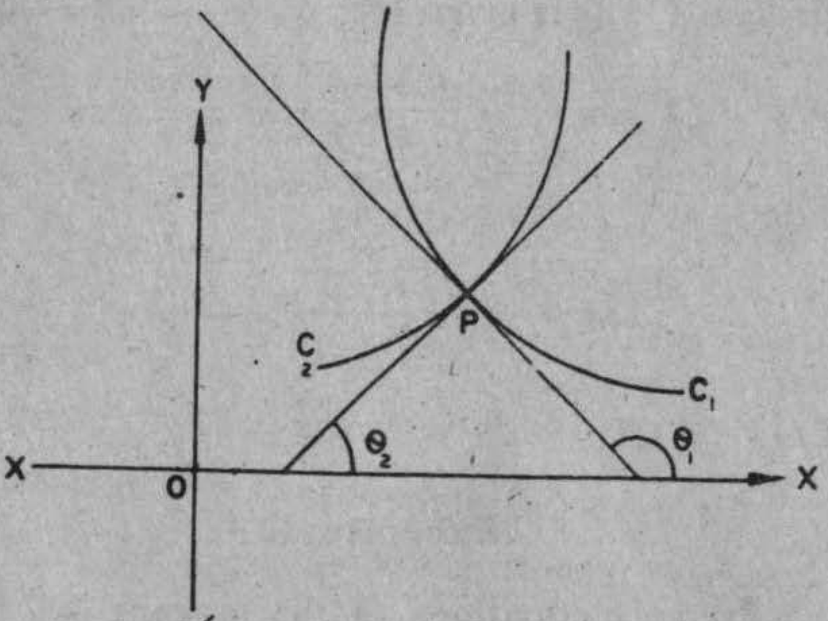


Fig. 10.3

The angle between the two curves at P is defined as the angle between their tangents at P.

The curves are said to intersect orthogonally if the angle between them is a right angle.

Example. Find the angle between the curves

$$y=x^2-3x+2 \text{ and } y=2x^2.$$

Solution. Subtracting the first equation from the second we have

$$x^2-3x+2=0$$
$$\Rightarrow (x-1)(x-2)=0$$
$$\therefore x=1,\ 2 \text{ and } y=2,\ 8.$$

Thus the points of intersection are (1, 2) and (2, 8)

Also $f_1'(x)=2x+3$ and $f_2'(x)=4x$,

so that at $x=1$, $f_1'(x)=5$ and $f_2'(x)=4$.

Again at $x=2$, $f_1'(x)=7$ and $f_2'(x)=8$.

$\therefore$ angle θ_1 at (1, 2) is given by

$$\text{Tan } \theta_1 = \frac{5-4}{1+5\times4}=\frac{1}{21}$$

and angle θ_2 at (2, 8) is given by

$$\text{Tan } \theta_2 = \frac{7-8}{1+7\times8}=-\frac{1}{57}.$$

Exercise 10·1

1. Find the equations of the tangent and the normal to the following curves at the given points on them :—

(*i*) $y=x^2-x+1$, $x=-1$
(*ii*) $y=x^2-2x$ $x=1$
(*iii*) $x^2=4y$ at (4, 4)
(*iv*) $y=x^2$ at (2, 4)
(*v*) $(x-3)^2+(y+2)^2=25$ at (6, 2)
(*vi*) $x^2+9y^2=225$ at (9, 4)
(*vii*) $x^2+4y^2-2x+8y=35$ at (3, 2)
(*viii*) $4x^2-25y^2+24x+50y+22=0$ at $(-1, 2)$
(*ix*) $x^2+y^2=a^2$ at (x_1, y_1)
(*x*) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1)
(*xi*) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1)
(*xii*) $y^2=4ax$ at (x_1, y_1).

2. Find the points on the curves where the tangents are parallel to the lines given below.

(*i*) $y=2x^2+7x$, $7x-y+9=0$

(*ii*) $y=54x^2-9x+34$, $36x+4y-17=0$

(*iii*) $y=\frac{1}{3}x^3-\frac{5}{2}x^2+2x+9$, $4x-y+33=0$.

3. Find the angle between each of the following pairs of curves.

(*i*) $y=x^2$, $y=2x^2-3x+2$

(*ii*) $y=2-x^2$, $y=2x^2-1$

(*iii*) $y=x^2-3x$, $y=x^2-25x+22$

(*iv*) $y=x^2+2x-1$, $y=2x^2$.

4. Prove that the curves $x^2+3y^2=24$ and $3x^2-y^2=12$ intersect at right angles at the point $(\sqrt{5}, \sqrt{6})$.

5. Find the angle at which the curves $y^2=ax$ and $x^2=ay$ intersect.

6. Find the angle of intersection of the curves

$$x^2-y^2=a^2,\ x^2+y^2=a^2\sqrt{2}.$$

7. Find the condition that the conics $ax^2+by^2=1$ and $a x^2+b'y^2=1$ should cut orthogonally.

ANSWERS

1 (*i*) $y-3=-3(x+1)$, $y-3=\frac{1}{3}(x+1)$

(*ii*) $y+1=0$, $x-1=0$

(*iii*) $2x-y-4=0$, $x+2y=12$

(*iv*) $4x-y-4=0$, $x+4y=18$

(*v*) $3x+4y-26=0$, $3y-4x=-18$

(*vi*) $x+4y=25$, $4x-y=32$

(*vii*) $x-3y+3=0$, $3x+y-11=0$

(*viii*) $8x-25y+58=0,\ 8y+25x+9=0$

(*ix*) $xx_1+yy_1=a^2,\ y_1x-x_1y=0$

(*x*) $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,\ a^2y_1x-b^2x_1y=(a^2-b^2)\,x_1\,y_1$

(*xi*) $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1,\ a^2y_1x+b^2x_1y=(a^2+b^2)\,x_1y_1$

(*xii*) $yy_1=2a\,(x+x_1),\ y_1x+2ay=y_1\,(x_1+2a)$.

2. (*i*) $(0, 0)$

(*ii*) $(0, 34)$

(*iii*) $x=\frac{5\pm\sqrt{33}}{2}$

3. (*i*) At $(1, 1)$, $\theta=\tan^{-1}\frac{1}{3}$; at $(2, 4)$, $\theta=\tan^{-1}\frac{1}{21}$

(*ii*) $\theta=\tan^{-1}\left(\pm\frac{8}{7}\right)$

(*iii*) At $(1, -2)$, $\theta=\tan^{-1}\frac{11}{12}$

(*iv*) 0, curves touch.

5. $\tan^{-1}\frac{3}{4}$

6. $135°, 45°$.

7. $\frac{a-a'}{aa'}=\frac{b-b'}{ab'}$.

10.5. *Increasing and Decreasing Functions*

A function $f(x)$ is said to be increasing at $x=a$ if for h, positive and sufficiently small

$$f(a-h)<f(a)<f(a+h).$$

A function $f(x)$ is said to be decreasing at $x=a$ if for h, positive and sufficiently small,

$$f(a-h)>f(a)>f(a+h).$$

A non-constant function is said to be an increasing (decreasing) function over an interval if it is increasing (decreasing) at every point of the interval.

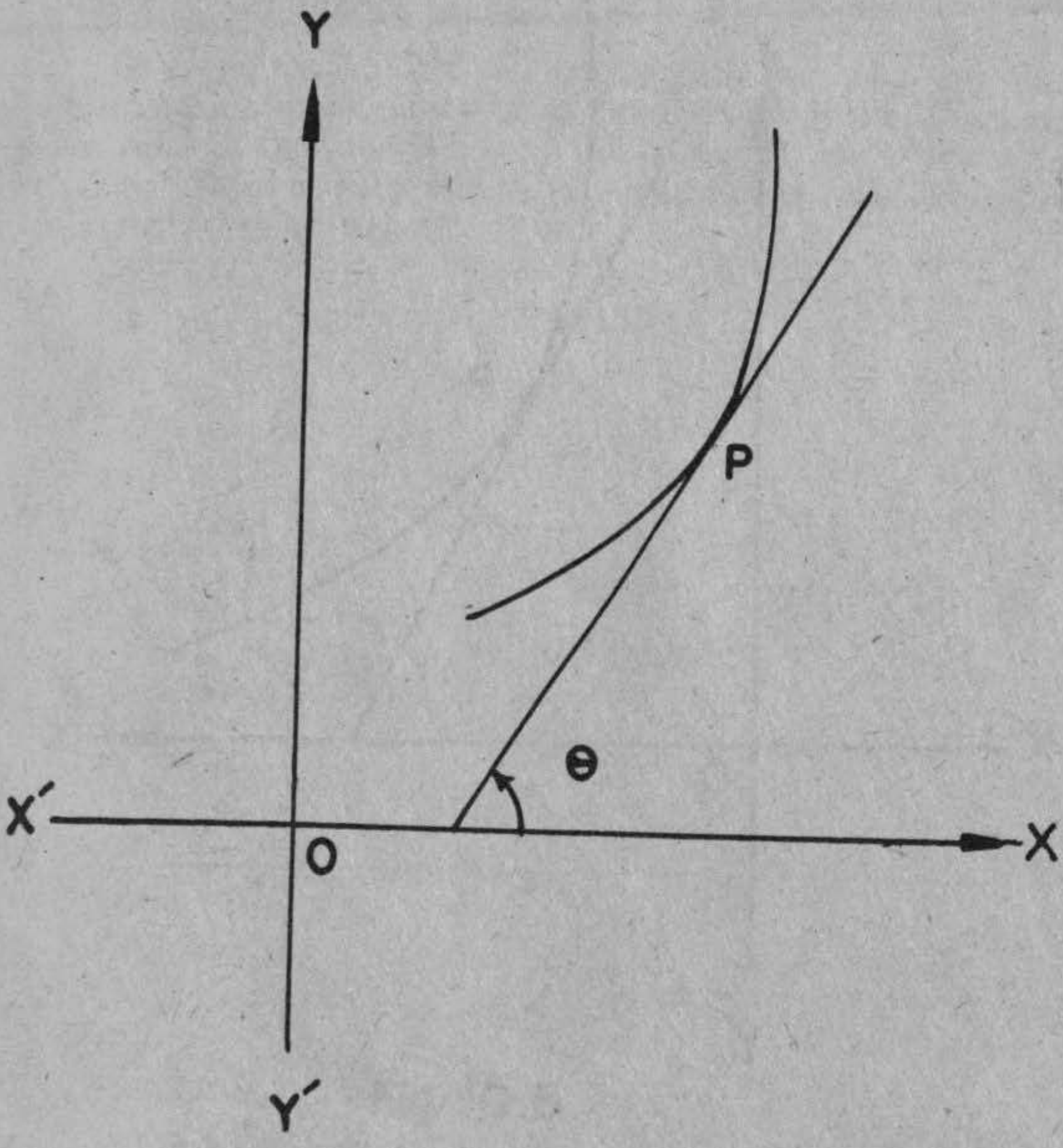


Fig. 10.4

***Criteria** to Determine Whether $f(x)$ is Increasing or Decreaing.* In figure 10 4 the inclination of the tangent PT at the point P, where the function is increasing is an acute angle, therefore, the slope of the tangent is positive.

Thus for an increasing function $f'(x)>0$ at the point.

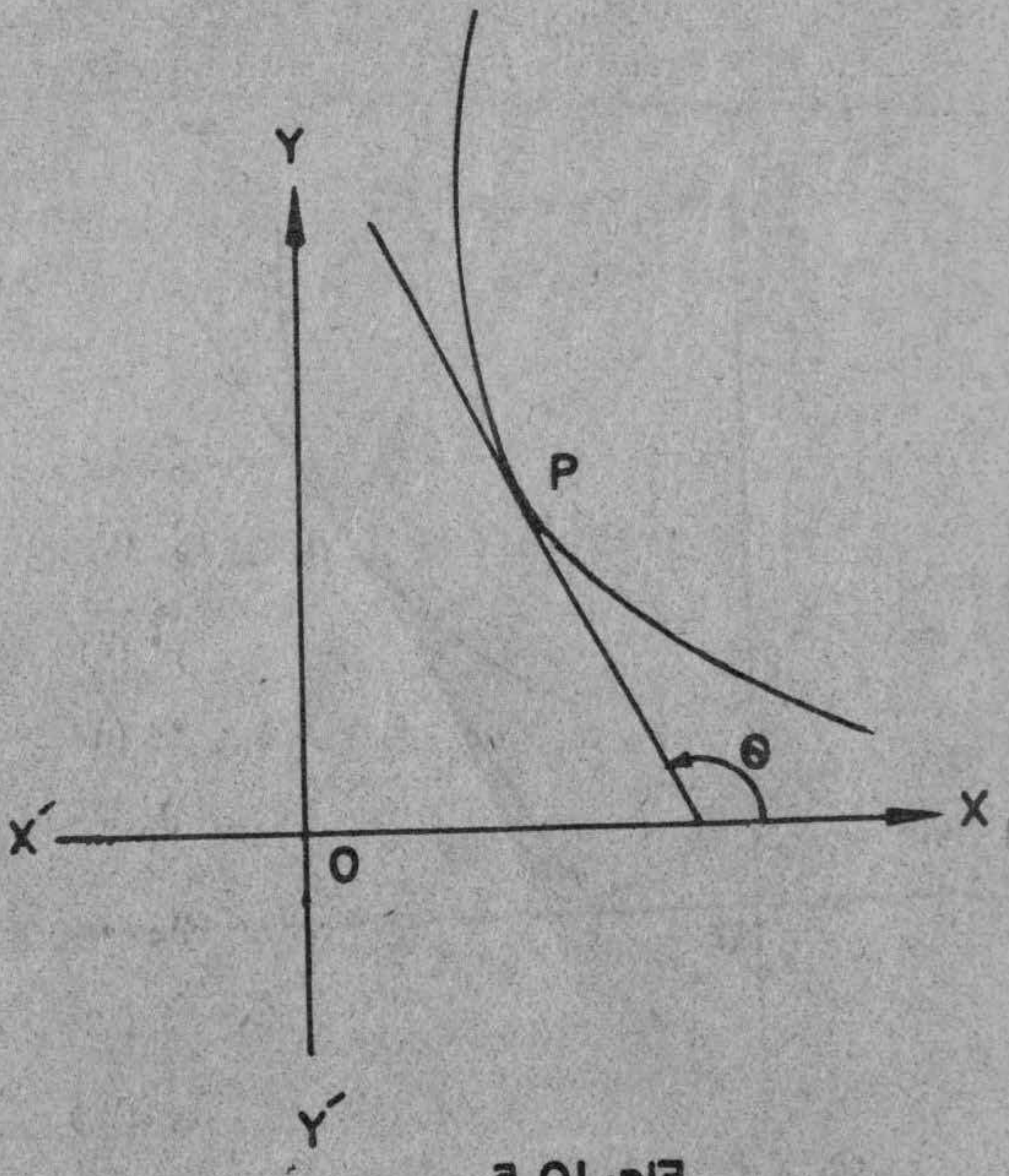


Fig. 10.5

In figure 10.5 the inclination of the tangent PT at the point P, where the function is decreasing is an obtuse angle, therefore the slope of the tangent is negative.

Thus for a decreasing fuction $f'(x) < 0$ at the point.

10.6. *Maxima and Minima of a Function*

Consider the graph of a function $y=f(x)$ which is continuous in an interval (a, b).

The points A, C, E where the function changes from an increasing to a decreasing one are called *points of maximum value* of the function, and the points B, D, F where it changes from a decreasing to an increasing one are called *points of minimum value*.

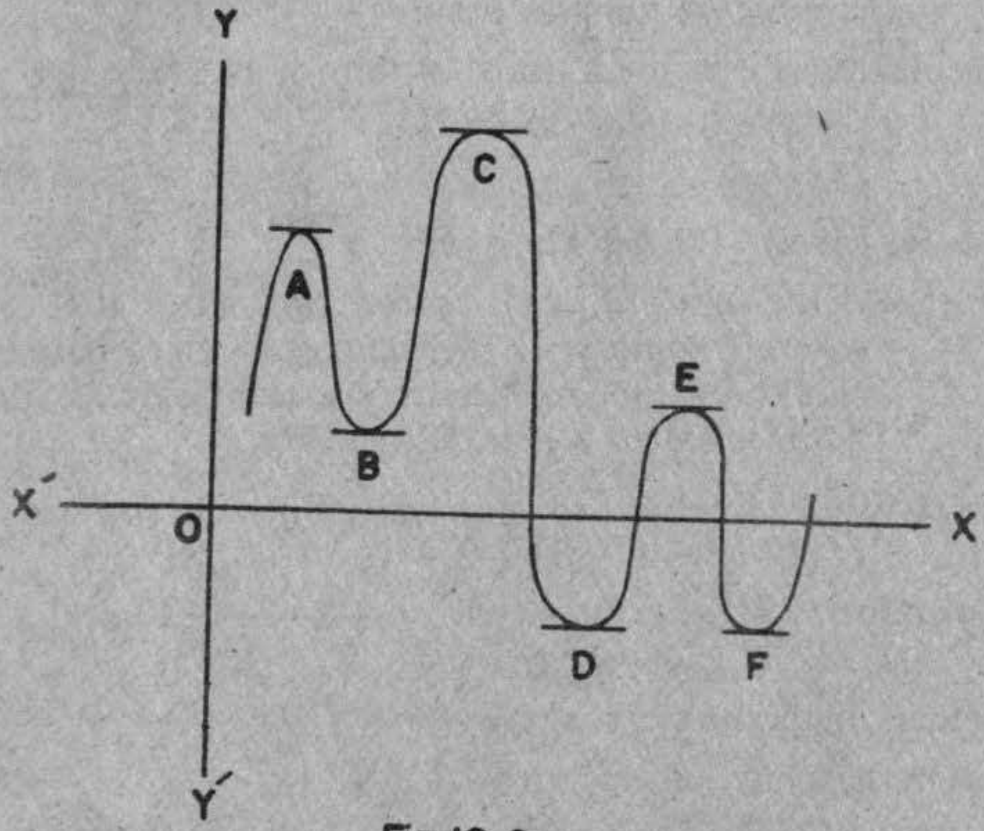


Fig. 10.6

The *maximum* and the *minimum points* taken together are called turning points or points of extreme values of the function.

The tangents to the curve $y = f(x)$ at its points of extreme value are parallel to x—axis.

Thus, at points of extreme value $f'(x)=0$. Hence to find points of extreme value of $f(x)$ we have to find the value of x for which $f'(x)$ vanishes by solving the equation.

$$f'(x)=0.$$

10.7. *How to Distinguish Between Maxima and Minima*

From the definition of a point of maximum value the function $f(x)$ is increasing before the point and decreasing after it.

Thus criteria for a point of maximum value are :—

$f'(x)>0$ before the point,
$f'(x)=0$ at the point,
$f'(x)<0$ after the point.

Similarly criteria for a point of minimum value are :-

$f'(x)<0$ before the point,
$f'(x)=0$ at the point,
$f'(x)>0$ after the point.

Note—If the sign of $f'(x)$ before and after the point where it vanishes is the same then there is neither maximum nor minimum at the point. Such a point is called a *point of inflexion.*

Example. Examine for extreme values the function

$$f(x)=x^3-x^2-8x+2.$$

Solution. Here $f'(x)=0$ at $x=-\frac{4}{3}=-1.33$ and $x=2$. Let $x=-1.33$.

Before the point say for $x=-1.4, f'(x)>0$.

After the point say for $x=-1.3, f'(x)<0$

$\therefore$ $x=1\cdot33$ is a point of max. value of the function and the maximum value of $f(x)$

$$=(-\tfrac{4}{3})^3-(\tfrac{4}{3})^2-8(-\tfrac{4}{3})+2$$
$$=8\tfrac{14}{27}=8.518 \text{ approx.}$$

Now consider $x=2$.

Before the point say for $x = 1.9$, $f'(x) < 0$

After the point say for $x = 2.1$, $f'(x) > 0$.

So that $x = 2$ is a point of minimum value of the function. The minimum value $= (2)^3 - (2)^2 - 8.2 + 2 = 8 - 4 - 16 + 2 = -10$.

10.8. *Procedure for Finding Maximum and Minimum Values*

From above it is clear that for finding points of maximum and minimum values we have to adopt the following procedure :—

(1) Find the points where $f'(x) = 0$.

(2) Examine the sign of $f'(x)$ before and after these points.

(3) Draw the conclusion according to criteria in § 10.7

(4) Substitute the values of x found in (1) in the function to find its maximum or minimum values as the case may be.

Example. Examine the function $f(x) = x^4 - 8x^2 + 10$ for extreme values.

Solution.
$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$$
$$= 4x(x+2)(x-2)$$
$$f'(x) = 0 \Rightarrow x = -2, 0, 2.$$

Before $x = -2$, $f'(x) < 0$, after $x = -2$, $f'(x) > 0$,

$\therefore x = -2$ is a point of minimum value, and minimum value $= -6$.

Before $x = 0$, $f'(x) > 0$, after $x = 0$, $f'(x) < 0$,

$\therefore x = 0$ is a point of maximum value,a nd maximum value $= 10$.

Before $x = 2$, $f'(x) < 0$, after $x = 2$, $f'(x) > 0$,

$\therefore$ $x = 2$ is a point of minimum value, and minimum value $= -6$

10.9. *Use of Second Derivative for Distinguishing Between Maxima and Minima of a Function*

Since for a point of maximum value

$f'(x)>0$ before the point,
$f'(x)=0$ at the point and
$f'(x)<0$ after the point.

$f'(x)$ is a decreasing function at the point and its derivative $f''(x)<0$ for a point of maximum value. Similarly for a point of minimum value $f'(x)$ is increasing and $f''(x)>0$

The complete discussion is beyond the scope of this book.

Example 1. In the example on page 413

$f''(x)=12x^2-16$, so that
$f''(-2)>0 \quad \therefore\ f(-2)$ is minimum ;
$f''(0)<0 \quad \therefore\ f(0)$ is maximum and
$f''(2)>0 \quad \therefore\ f(2)$ is minimum.

Example 2. Discuss for maxima and minima the function

$$(x-2)^3(x-3)^2.$$

Solution. Here $f(x)=(x-2)^3(x-3)^2$

$$\therefore\ f'(x)=(x-2)^3\,2(x-3)+3(x-2)^2(x-3)^2$$
$$=(x-2)^2(x-3)(5x-13).$$
$$f''(x)=5(x-2)^2(x-3)+(x-2)^2(5x-13)$$
$$+2(x-2)(x-3)(5x-13).$$

Now $f'(x)=0$ for $x=2, \frac{13}{5}, 3$.

For $x=2$, $f''(2)=0$, hence second derivative cannot give us any conclusion.

$f'(x)$ before 2 (at $x=1.9$, say) is positive,
$f'(x)$ after 2 (at $x=2.1$, say) is also positive.
$\therefore\ x=2$ is a point of inflexion.

For $x=\frac{13}{5}$, $f''\left(\frac{13}{5}\right)$ is negative, $\therefore\ f(x)$ is maximum at

$x = \frac{13}{5}$, and maximum value of $f(x) = \left(\frac{3}{5}\right)^3 \left(\frac{-2}{5}\right)^2 = \frac{108}{3125}$.

For $x=3$, $f''(x)$ is positive $\therefore f(x)$ is minimum at $x=3$, and minimum value of $f(x)=0$.

Example 3. Discuss the function $\text{Sin } x+\text{Cos } 2x$ for extreme values in the interval $(0, 2\pi)$.

Solution. Let $y=\text{Sin } x+\text{Cos } 2x$, ... (1)

so that $\frac{dy}{dx}=\text{Cos } x-2 \text{ Sin } 2x$

$=\text{Cos } x-4 \text{ Sin } x \text{ Cos } x.$

Now $\frac{dy}{dx}=0 \Rightarrow \text{Cos } x\,(1-4 \text{ Sin } x)=0$

$\Rightarrow \text{Cos } x=0, \text{ Sin } x=\frac{1}{4}$

$\Rightarrow x=\frac{\pi}{2}, \frac{3\pi}{2}$ and $x=\text{Sin}^{-1}\frac{1}{4}, \pi-\text{Sin}^{-1}\frac{1}{4}$.

We have $\frac{d^2y}{dx^2}=-\text{Sin } x-4 \text{ Cos } 2x.$

For $x=\frac{\pi}{2}$, $\frac{d^2y}{dx^2}=3$ (positive).

Also for $x=\frac{3\pi}{2}$, $\frac{d^2y}{dx^2}=5$ (positive)

Hence y is *a* minimum for $x=\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Again $\frac{d^2y}{dx^2}=-\text{Sin } x-4\,(1-2 \text{ Sin}^2x)$ and so for

$=\text{Sin}^{-1}\frac{1}{4}$ and $\pi-\text{Sin}^{-1}\frac{1}{4}$

$\frac{d^2y}{dx^2}=-\frac{1}{4}-4\,(1-2.\,\frac{1}{16})=-\frac{15}{4}$, which is negative

Therefor y is a maximum for $x=\text{Sin}^{-1}\frac{1}{4}$ and $\pi-\text{Sin}^{-1}\frac{1}{4}$.

Putting the values of x in (1), we see that $0, -2$ are the two minimum values, while $\frac{9}{8}$ is the maximum value.

Example 4. Show that x^x is a minimum for

$$x=\frac{1}{e}.$$

Solution. Let $y=x^x$, $\ln y = x \ln x$.

$$\therefore \quad \frac{1}{y}\ \frac{dy}{dx} = 1+\ln x$$

$$\text{or} \quad \frac{dy}{dx} = x^x\,(1+\ln x).$$

$$\text{Now} \quad \frac{dy}{dx} = 0 \Rightarrow x^x\,(1+\ln x)=0$$

$$\Rightarrow 1+\ln x = 0 \ (\because\ x^x \neq 0)$$

$$\Rightarrow \ln x = -1$$

$$\Rightarrow x = \frac{1}{e}.$$

$$\text{Now} \quad \frac{d^2y}{dx^2} = x^x\left(\frac{1}{x}\right)+(1+\ln x)\ \ x^x\,(1+\ln x)$$

$$= x^x\left(\frac{1}{x}\right)+x^x\,(1+\ln x)^2.$$

$\dfrac{d^2y}{dx^2}$ at $x=\dfrac{1}{e}=\left(\dfrac{1}{e}\right)^{\frac{1}{e}}(e)$, which is positive and hence x^x is a minimum for $x=\dfrac{1}{e}$.

Exercise 10.2

1. Examine for extreme values the following functions :—

(*i*) $x^3-7x^2+8x+32$

(*ii*) $(\frac{1}{3})\ x^3-2x^2+3x+1$

(*iii*) $x^5-4x^4+4x^3$.

2. Find the maximum and minimum values of the following :—

(*i*) $\dfrac{x^2}{(x+2)(x+3)}$

(*ii*) $\frac{(x-1)(x-6)}{x-10}$.

(*iii*) $\frac{x^3}{2x+4}$.

3. Discuss for maxima and minima the following :—

(*i*) $(x-2)^3(x-3)^2$

(*ii*) $(x-4)^2(x-2)$

(*iii*) $(x-1)(x-2)^2$.

4. Find the point of inflexion of the curve

$$a^2y=(x-b)^3.$$

5. Discuss for extreme values the following functions :—

(*i*) $\text{Sin } x+\text{Cos } x$

(*ii*) $\text{Sin } x \text{ Cos}^2 x$

(*iii*) $\frac{\text{Sin } x \text{ Cos } x}{\text{Sin } x+\text{Cos} x}$

(*iv*) $a \text{ Sin } x+b \text{ Cos } x$.

6. Show that maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$

7. Prove that the maximum value of $\frac{\ln x}{x}$ is $\frac{1}{e}$.

ANSWERS

1. (*i*) Max. value $\frac{644}{27}$, Min. value 16

(*ii*) Max. value $\frac{7}{3}$. Min. value 1

(*iii*) Max. value $\frac{3456}{3125}$, Min value 0

2. (*i*) Max. value -24, Min. value 0

 (*ii*) Max. value 1, Min. value 25

 (*iii*) Min. value $13\frac{1}{2}$

3. (*i*) Max. value $\frac{108}{3125}$, Min. value 0

 (*ii*) Max. value $\frac{32}{27}$, Min. value 0

 (*iii*) Max. value $\frac{4}{27}$, Min. value 0

4. $(b, 0)$

5. (*i*) Max. $\sqrt{2}$, Min. $\sqrt{2}$

 (*ii*) Max $\frac{3\sqrt{3}}{16}$, Min. $-\frac{3\sqrt{3}}{16}$.

 (*iii*) Max. $\frac{1}{2\sqrt{2}}$, Min. $-\frac{1}{2\sqrt{2}}$

 (*iv*) Max. $\sqrt{a^2-b^2}$, Min. $-\sqrt{a^2+b^2}$

10.10. *Problems in Maxima and Minima*

Example 1. A rectangle has a fixed perimeter p. Show that its area will be maximum if it is a square.

Solution. Let x and y be the length and breadth of the rectangle. Then

$$\text{perimeter} = 2x+2y = p \text{ or } y = \frac{p-2x}{2},$$

$$\therefore \quad \text{area} = xy = x\,\frac{(p-2x)}{2} = \frac{px}{2} - x^2 = f(x).$$

$$f'(x) = \frac{p}{2} - 2x,\ f''(x) = -2.$$

Now $f'(x) = 0$ gives $x = \frac{p}{4}$ and then $y = \dfrac{p-\frac{p}{2}}{2} = \frac{p}{4}$.

Since, $f''(x)$ is negative, $f(x)$ is maximum when length and breadth of the rectangle are equal *i.e.* it is a square.

$$\text{Maximum area} = \frac{p}{4} \cdot \frac{p}{4} = \frac{p^2}{16}.$$

Example 2. An open rectangular box has a square base and the combined area of the walls and the base is 48 square feet. Find the dimensions of the box when its volume is maximum.

Solution. Let the length, breadth and height of the box be x, y and y.

$$\text{Area of walls and bottom} = 4xy + x^2 = 48$$

$$\text{or } y = \frac{48 - x^2}{4x} \tag{1}$$

$$\text{Volume} = x^2 y = x^2 \frac{48 - x^2}{4x} = 12x - \tfrac{1}{4} x^3 = f(x).$$

$$f'(x) = 12 - \tfrac{3}{4}x^2,$$

$$f''(x) = -\tfrac{3}{2} x.$$

$$\therefore f'(x) = 0 \text{ gives } x = 4, -4$$

Now negative value for length or breadth is not permissible. When $x=4$, $f''(x) = -6$ *i.e.* negative. Hence volume is maximum when $x=4$. Putting $x=4$ in (1) we get $y=2$.

Exercise 10.3

1. Express the number 4 as the sum of two positive numbers in such a way that the sum of the square of the first and the cube of the second is as small as possible.

2. The perimeter of a rectangle is 100 feet, find the sides when the area is maximum.

3. Divide 20 into two parts so that the sum of their squares is the least.

4. Find the positive number such that the sum of its square and 16 times the square of its reciprocal is as small as possible.

5. What number added to its reciprocal gives the least possible sum ?

6. A farmer wishes to fence off a rectangular pasture along a straight river, the side along the river requiring no fence He has barbed wire enough to build a fence one kilometers long. What is the area of the largest pasture of the above description which he can fence ?

7. A rectangle is to have an area of 64 sq. centimeters. Find its dimensions so that the distance from one corner to the mid. point of a non-adjacent side be a minimum.

8. An open rectangular box is to be made from a piece of cardboard 8 centimeters wide and 15 centimeters long by cutting a square from each corner and bending up the sides. Find the dimensions of the box of largest volume.

9. The amount of fuel consumed per hour by a certain steamer varies as the cube of its speed. When the speed is 15 kilometer per hour the fuel consumed is $4\frac{1}{2}$ tonnes of coal per hour at Rs. 4 per tone. The other expenses total Rs. 100 per hour. Find the most economical speed and the cost of a voyage of 1980 kilometers.

10. A square sheet of tin '*a*' centimeters on a side is to be used to make an open-top box by cutting a small square of tin from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have as large a volume as possible.

11. If 40 square decimeters of sheet metal are to be used in the construction of an open tank with a square base, find the dimensions so that the capacity is greatest possible.

12. The maker of a certain article finds that, in order to sell x articles each week, he must price them at

Rs. $\sqrt{900-2x^2}$ a piece. What number of articles per week will bring him the maximum possible revenue ?

13. The rate Q at which water flows over a certain spillway is proportional to $D(H-D)^{\frac{1}{2}}$, where D is the depth of the flow and H is the head. For a fixed value of H, what value of D makes Q a maximum ?

ANSWERS

1. $\frac{4}{3}, \frac{8}{3}$
2. Each side = 25 feet
3. 10, 10
4. 2
5. 1
6. $\frac{1}{8}$ square mile
7. $8\sqrt{2}$, $4\sqrt{2}$ inches
8. $\frac{5}{3}, \frac{14}{3}, \frac{35}{3}$ (inches)
9. $15\left(\frac{5}{3}\right)^{\frac{2}{3}}$ m.p.h, $19800\left(\frac{3}{5}\right)^{\frac{2}{3}}$ rupees
10. Each side is $\frac{a}{6}$
11. $\sqrt{\frac{40}{3}}, \sqrt{\frac{40}{3}}, \frac{1}{2}\sqrt{\frac{40}{3}}$ (feet)
12. 15
13. $D=\frac{2H}{3}$

10.11. *Motion in a Straight Line*

The motion of a P along a particle straight line is completely described by the equation $s=f(t)$, where $t \geqslant 0$ is time and s is distance of P from a fixed point O in its path.

The velocity of P at time t is $v\frac{ds}{dt}$.

If $v > 0$, P is moving in the direction of increasing s.

If $v < 0$, P is moving in the direction of decreasing s.

If $v = 0$, P is instantaneously at rest.

The acceleration of P at time t is given by

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

If $a > 0$, v is increasing but if $a < 0$, v is decreasing.

If v and a have opposite signs, the speed of P is decreasing.

Example 1. A body moves along a straight line according to the law $s = \frac{1}{2}t^3 - 2t$. Find its velocity and acceleration at $t = 2$.

Solution. $s = \frac{1}{2}t^3 - 2t$

$$\Rightarrow v = \frac{ds}{dt} = \frac{3}{2}t^2 - 2.$$

When $t = 2$, $v = (\frac{3}{2})(2)^2 - 2 = 4$ ft./sec.

Now $a = \dfrac{dv}{dt} = 3t$.

When $t = 2$, $a = 3(2) = 6$ ft./sec.^{2}

Example 2. The path of a particle moving in a straight line is given by $s = t^3 - 6t^2 + 9t + 4$,

(*i*) find s and a when $v = 0$

(*ii*) find s and v when $a = 0$

(*iii*) when is s increasing ?

(*iv*) when is s increasing ?

(*v*) when does the direction of motion change ?

Solution.

$$v = \frac{ds}{dt} = 3t^2 - 12t + 9 = 3(t-1)(t-3),$$

$$a = \frac{dv}{dt} = 6(t-2).$$

(*i*) $v=0 \Rightarrow t=1, t=3$.

When $t=1$, $s=8$ and $a=-6$.
When $t=3$, $s=4$ and $a=6$.

(*ii*) $a=0 \Rightarrow t=2$.

When $t=2$, $s=6$ and $v=-3$.

(*iii*) s increases when $v>0$, *i e.* when $t<1$ and $t>3$

(*iv*) v increases when $a>0$, *i.e.* when $t>2$.

(*v*) the direction of motion changes when $v=0$ and $a \neq 0$. From (*i*) the direction changes when $t=1$ and $t=3$.

Example 3. The position of a point P on **x-axis** at time t is given by $s(t)=2t^3-9t^2+20$. Discuss the motion by finding the velocity $v(t)$ and the acceleration $a(t)$ at

$$t=0,\ 1,\ 2,\ 3,\ 4 \text{ and } 5 \text{ secs.}$$

Solution. $s(t)=2t^3-9t^2+20$

$$v(t)=\frac{ds}{dt}=6t^2-18t=6t\,(t-3)$$

$$a(t)=\frac{dv}{dt}=12t-18.$$

The values of velocity and acceleration at the required times are given below :—

t	s(t)	v(t)	a(t)
0	20	0	-18
1	13	-12	-6
2	0	-12	6
3	-7	0	18
4	4	24	30
5	45	60	42

The table shows that at time $t=0$, the body is 20 feet to the right of 0 and its velocity is 0. At $t=1$, it is at a distance of 13 feet to the right of 0 and is moving towards left at 12 ft/sec At $t=2$, it is at 0 and still moving towards left at 12 ft/sec. At $t=3$, it is 7 feet to the left of 0 and then velocity is zero, showing a momentary stop at that instant. After that the velocity and the distance begin to increase in the direction from left to right. The values of acceleration at various instants give an indication whether the velocity at a particular instant is decreasing or increasing.

Exercise 10.4

· A particle moves in a straight line according to the law $s=t^3-6t^2+9t$, the units being feet and seconds. Locate the particle with respect to its initial position $(t=0)$ at 0; find its direction and velocity and determine whether its speed is increasing or decreasing when

(a) $t=\frac{1}{2}$, (b) $t=\frac{3}{2}$, (c) $t=\frac{5}{2}$, (d) $t=4$.

2. The path of a particle moving in a straight line is given by

(i) $s=t^3-9t^2+24t$
(ii) $s=t^3-3t^2+3t+3$
(iii) $s=2t^3-12t^2+18t-5$
(iv) $s=3t^4-28t^3+90t^2-108t$.

Find in each case

(a) s and a when $v=0$,
(b) s and v when $a=0$,
(c) when is s increasing ?
(d) when is v increasing ?
(e) when does the direction of motion change ?

3. A body moves along a horizantal line according to the law $s=f(t)=t^3-9t^2+24t$.

(*i*) when is s increasing and when decreasing ?

(*ii*) when is v increasing and when decreasing ?

(*iii*) when is the speed of the body increasing and when decreasing ?

(*iv*) Find the total distance travelled in the first 5 seconds of motion.

4. A particle moves in a horizontal line according to the law $s=f(t)=t^4-6t^3+12t^2-10t+3$

(*i*) when is the speed increasing and when decreasing ?

(*ii*) when does the direction of motion change ?

(*iii*) find the total distance travelled in the first 3 seconds of motion.

5. A body rises vertically from earth according to the law

$$s=64t-16t^2,$$

show that it has lost one-half its velocity in its first 48 feet of rise.

ANSWERS

1. (*a*) $\frac{25}{8}$ ft. to the right of 0 ; moving to the right with $v=\frac{15}{4}$ ft./sec ; decreasing.

(*b*) $\frac{27}{8}$ to the right of 0 ; moving to the left with $-\frac{9}{4}$ ft /sec ; decreasing.

(*c*) $\frac{5}{8}$ ft. to the right of 0 ; moving to the left with $v=-\frac{9}{4}$ ft./sec ; increasing.

(*d*) 4 ft. to the right of 0 ; moving to the right with $v=9$ ft./sec ; increasing.

2 (*a*) Stops at $t=2$ and $t=4$ with change of direction.

(*b*) Stops at $t=1$ without change of direction.

(c) Stops at $t=1$ and $t=3$ with change of direction.

(*d*) Stops at $t=1$ with change and at $t=3$ without change of direction.

3. (*i*) $t<2, t>4 ; 2<t<4$

(*ii*) Increasing : $t>3$, decreasing : $t<3$

(*iii*) Increasing : $2<t<3, t>4$

decreasing : $t<2, 3<t<4$

(*iv*) 28 ft.

4. (*i*) Increasing : $t<1$ and $t>2$

Decreasing : $1<t<2$

(*ii*) $t=2{\cdot}5$

(*iii*) $\frac{51}{8}$

10.12. *Related Rates*

Some times, while dealing with physical and geometrical problems, we come across two quantities which are so related that each can be considered as the function of the other and both as functions of time. In such problems, if we know the rate of change of one of the quantities with respect to time, we can find the rate of change of the other.

Example 1. A ladder 15 feet long rests against a house. It slides down, the lower end slipping along the level ground at the rate of 2 ft./sec. How fast is the upper end of the ladder sliding down the wall when it is 12 ft. from the ground ?

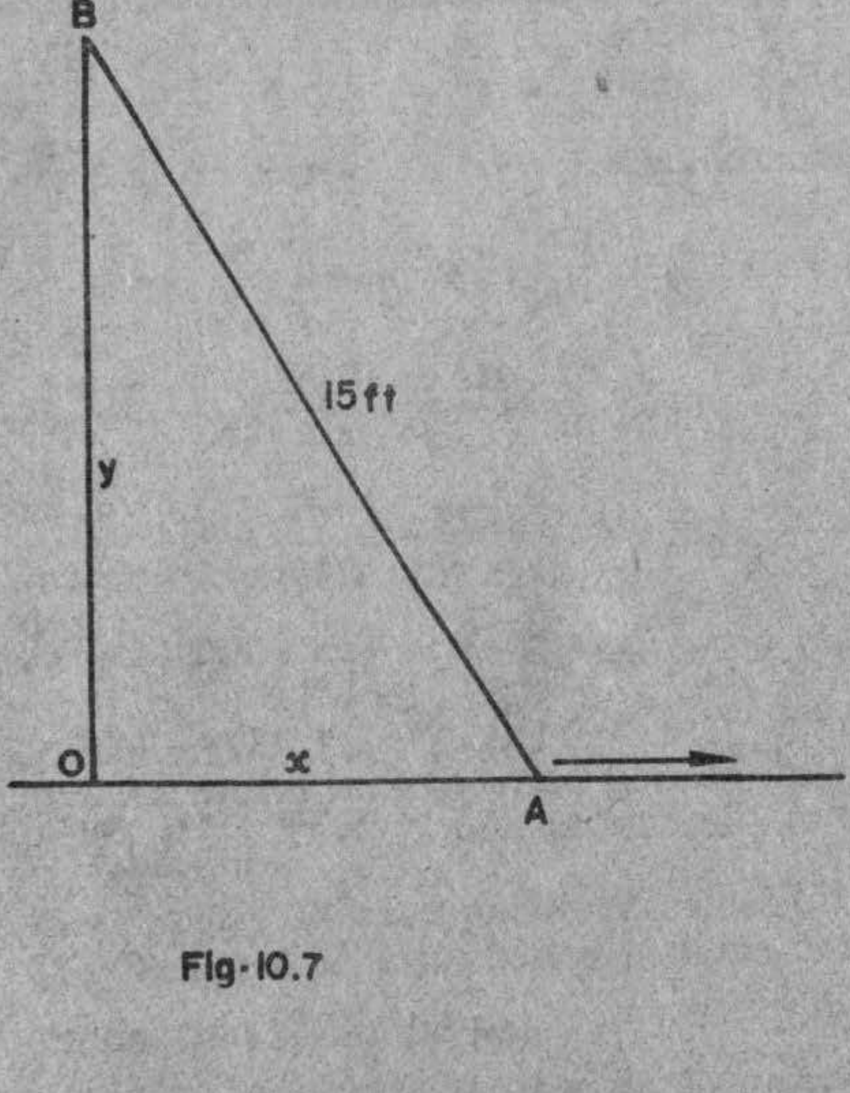


Fig-10.7

Solution. The figure shows the position of the ladder at time t. We have supposed the distance from the base of the house-wall to the foot of the ladder to be x and distance from the ground to the top of the ladder to be y because both these distances are variables.

when $y=12$ ft, $\frac{dx}{dt}=2$ ft./sec.

We have to find $\frac{dy}{dt}$ with the condition

$$x^2+y^2=225.$$

Differentiating w.r.t. 't'

$$2\,x\,\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\text{or}\quad \frac{dy}{dt}=-\frac{x}{y}\ \ \frac{dx}{dt}.$$

When $y=12$, $x=\sqrt{225-144}=9$, $\frac{dx}{dt}=2$,

$$\therefore\quad \frac{dy}{dt}=-\frac{9}{12}(2)=-\frac{3}{2}\,.$$

$\therefore$ y is decreasing (*i.e.* top of the ladder is moving down) at the rate of $\frac{3}{2}$ feet per second.

Such a problem involving rates of related variables is called a problem in " related rates ".
In such a problem,

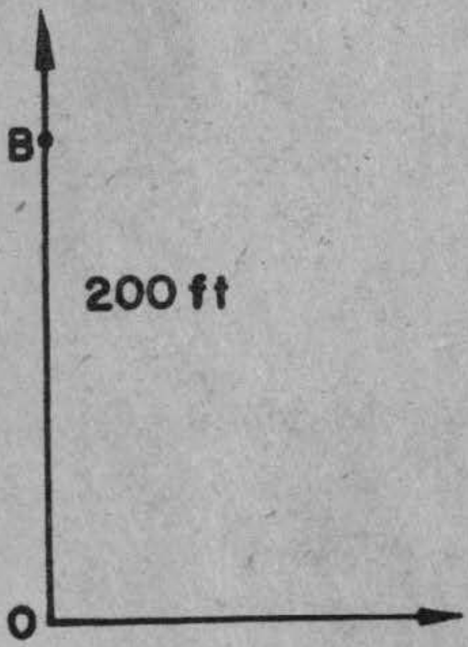
(*i*) certain variables are related in a definite way for all values of t under consideration,

(*ii*) the values of some or all of these variables and the rate of change of some of them are given at some particular instant, and

(*iii*) it is required to find the rate of change of one or more of them at this instant.

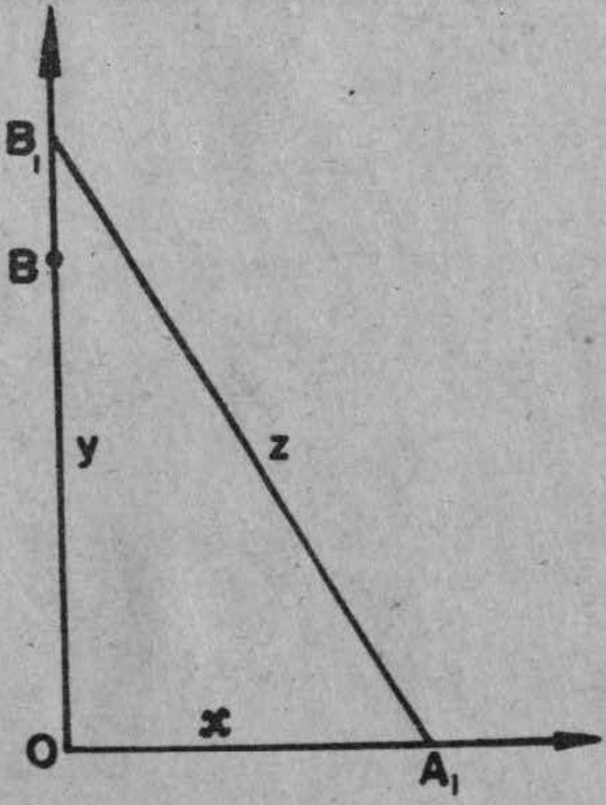
Example 2: A balloon is 200 d.m. off the ground and rising vertically at the constant rate of 15 d.m. per second.

An automobile passes beneath it travelling along a straight road at the constant rate of 66 ft/sec. How is the distance between them changing one second later ?



Initial Position

Fig.10.8



t seconds later

Fig.10.9

Solution. Let at any instant the automobile be at A_1 and the baloon be at B_1 such that $OA_1=x$ and $OB_1=y$.

Let distance between the two be z.

i e. $A_1B_1=z$

$$\therefore \quad z^2=x^2+y^2 \tag{1}$$

so that $2z\dfrac{dz}{dt}=2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}$

$$\text{or} \quad z\frac{dz}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt} \tag{2}$$

Now after 1 second,

$$x = 66 \text{ dm.}, y = 215 \text{ dm.}$$

and $z = \sqrt{(66)^2 + (215)^2} = 225$ dm. approximately.

$$\frac{dx}{dt} = 66, \frac{dy}{dt} = 15$$

$\therefore$ By (2),

$$225 \frac{dz}{dt} = (66).(66) + (215)(15)$$

or $\frac{dz}{dt} = 33.7$ dm./sec.

which is positive.

$\therefore$ distance between the automobile and the balloon is increasing at the rate of 33.7 dm./sec.

Exercise 10.5

1. Suppose a cone of height 12 centimeters is changing in shape due to the change in radius of base. What rate of increase of the radius will make the lateral area of the surface of cone increase at the rate of 10π square centimeters per minute when the radius of the base is 5 centimeters ?

2. A 26 decimeters long ladder leans against a vertical wall. At a particular instant, the foot of the ladder is 10 decimeters out from the base of the wall and is being drawn away from the wall at the rate of 4 dm./sec. How fast is the top of the ladder moving down the wall at this instant ?

3. A man is running over a bridge at a rate of 10 dm./sec. while a boat passes under the bridge and immediately below him at the rate of 20 decimeters per second. The boat's course is at right angles to the course of the man, and 20 decimeters below it. How fast are boat and man separating 1 second later ?

4. A man 6 feet tall walks at the rate of 5 ft/sec. towards a street light that is 16 feet above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 feet from the base of the light.

5. A bomber is in level flight at 8 miles above the ground. The flight path passes directly over a rocket installation. How fast is the bomber flying if the airline distance to the rocket installation is decreasing at 4 miles per minute and this distance is 10 miles?

ANSWERS

1. ·67 inches per second.
2. $\frac{5}{3}$ ft./sec. downward.
3. $\frac{50}{3}$ feet/second.
4. 8 ft./sec. towards the light; decreasing 3 ft./sec.
5. $\frac{20}{3}$ miles per minute towards the point directly above the rocket installation.

10.13. *Application of Integral Calculus to Geometry*

Example 1. The velocity v at time t in case of a moving object is given by

$$v(t)=3t^2+2t.$$

If the displacement at time t_0 is s_0, find s as a function of t.

Solution. $v(t)=\frac{ds}{dt}=3t^2+2t.$

$$s=\int(3t^2+2t)\,dt=t^3+t^2+c.$$

Putting $s=s_0$ and $t=t_0$ in the above relation, we get

$$s_0=t_0^3+t_0^2+C$$

$$\Rightarrow C=s_0-t_0^3-t_0^2.$$

$\therefore$ the distance s as a function of t is given by,

$$s-s_0=t^3+t^2-(t_0^3+t_0^2)$$

There is an indication in the above example that, given the velocity $v=\frac{ds}{dt}$ as a function of t, the distance s is given by the definite integral

$$s-s_0=\int_{t_0}^{t} v\,dt.$$

Example 2. A particle is moving in a straight line with acceleration $a=6t^2-36$.

Given that

$v=16$, $s=45.5$ when $t=0$, first v and s at $t=1,2,3,4,5,6$.

Solution. $\frac{dv}{dt}=6t^2-36$.

$\therefore\ v=2t^3-36t+C_1$.

Putting $v=16$, $t=0$ in the above relation, we get $C_1=16$.

$\therefore\ v=\frac{ds}{dt}=2t^3-36t+16$.

$\therefore\ s=\frac{1}{2}t^4-18t^2+16t+C_2$.

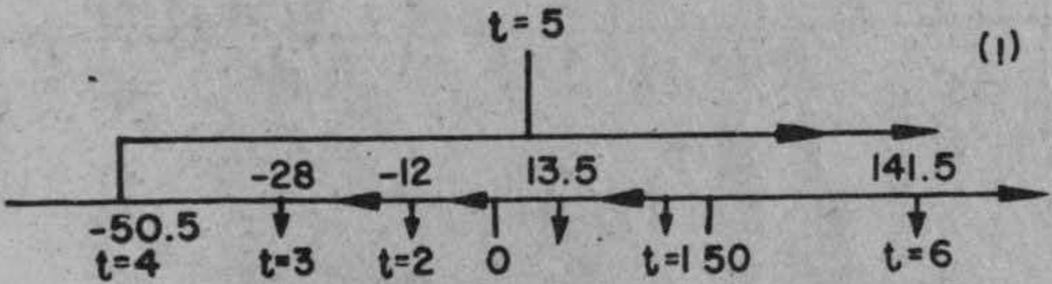
Putting $s=45.5$, $t=0$ in the above, we get $C_2=45.5$.

$$\therefore\ s=\tfrac{1}{2}t^4-18t^2+16t+45.5$$

The required values of v and s are shown in the table below.

t	v	s
1	-18	44
2	-40	13.5
3	-38	-28
4	0	-50.5
5	86	-12
6	232	141 5

The motion is shown in the following sketch.


t=5
(1)
-28
-12
13.5
141.5
-50.5
t=4
t=3
t=2
0
t=1
50
t=6

10.14. *Area Under a Curve*

Consider the are abounded by the continuous curve $y=f(x)$. the x-axis and the ordinates at P′ $(a, 0)$ and at P $(x, 0)$.

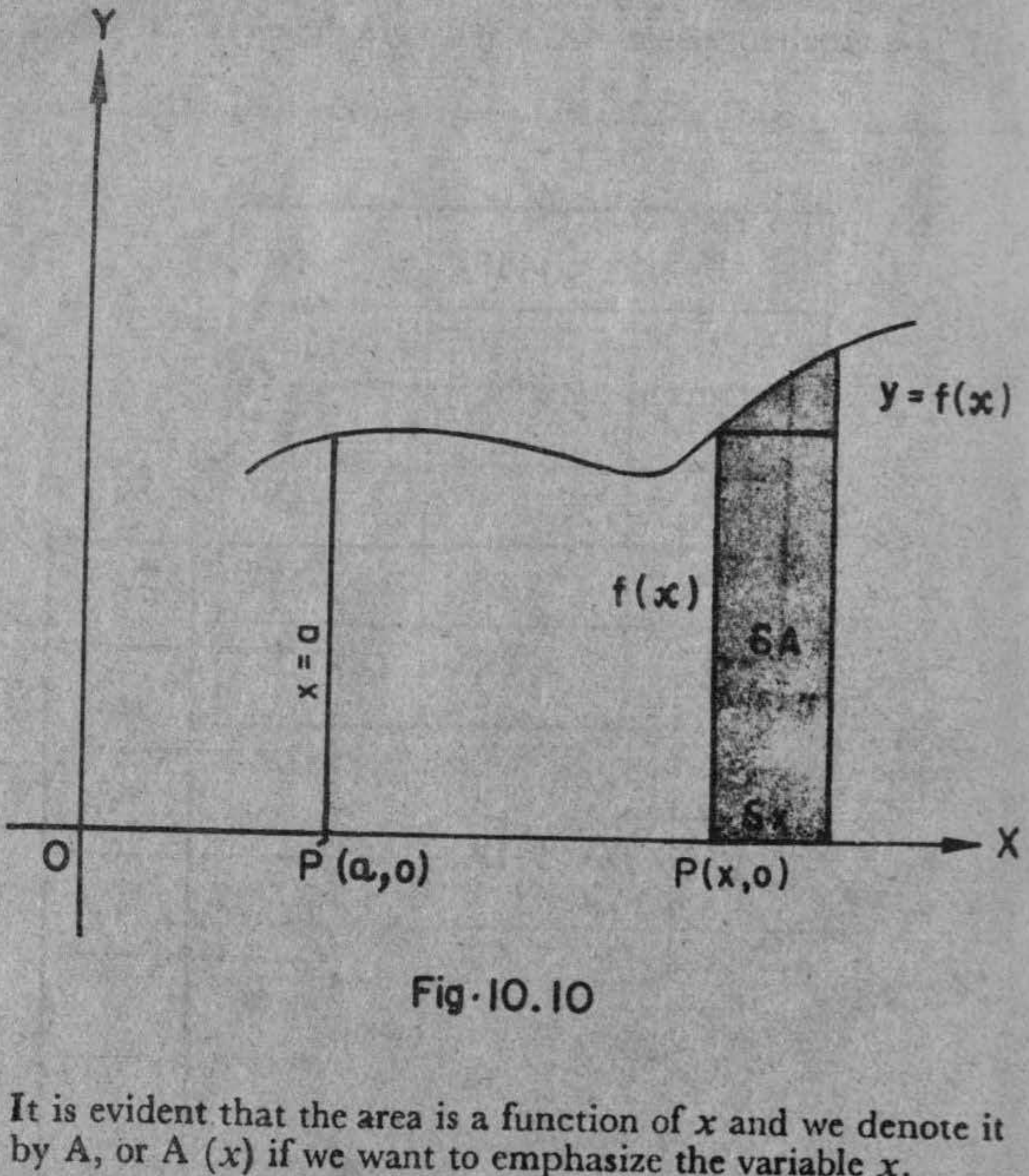


Fig. 10.10

It is evident that the area is a function of x and we denote it by A, or A (x) if we want to emphasize the variable x.

Let x increase by a small quantity δx, and let the corresponding increase in the area A be δA, which is shown by the shaded portion in the figure. The shaded portion is approximately equal to that of a rectangle of height $f(x)$ and base of width δx.

Hence $\delta A = f(x)\, \delta x$

$$\Rightarrow \frac{\delta A}{\delta x} = f(x).$$

Taking limit when $\delta x \to 0$, we get

$$\frac{dA}{dx} = f(x).$$

This gives $A = \int f(x)\, dx = F(x) + C$, where $F'(x) = f(x)$.

We know that $A = 0$ where $x = a$.

$$\therefore \quad 0 = F(a) + C \quad \text{or} \quad C = -F(a).$$

Hence $A(x) = F(x) - F(a) = \int_a^x f(x)\, dx$.

Example 1. Find the area enclosed between the curve $y = \frac{1}{4} x^2$ and the x-axis from $x = 0$ to $x = 4$.

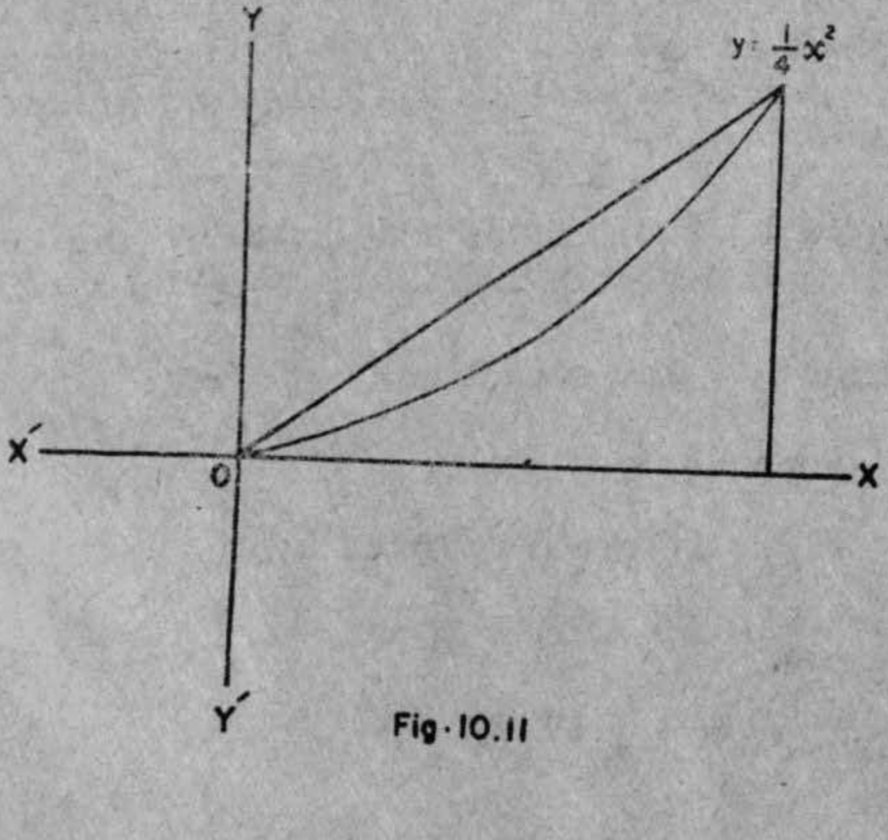


Fig. 10.11

Solution. Required area $= \int_0^4 \frac{1}{4} x^2 dx$

$$= \frac{1}{4}\int_0^4 x^2 dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{12}(64-0) = \frac{16}{3}.$$

Note : If f and g are continuous and $g(x) \leqslant f(x)$ for $a \leqslant x \leqslant b$, then the area of the set defined by $a \leqslant x \leqslant b$, $g(x) \leqslant y \leqslant f(x)$ is given by

$$\int_a^b [f(x) - g(x)]dx.$$

In order to determine the inequality $g(x) \leqslant f(x)$ in a problem, we find the sign of $g(x) - f(x)$ at some point between $x = a$ and $x = b$. If the sign is negative, we have the desired result.

Example 2. Find the area enclosed by the curves $y = x^2$, and $y = 3x$ between the ordinates $x = 1$ and $x = 2$.

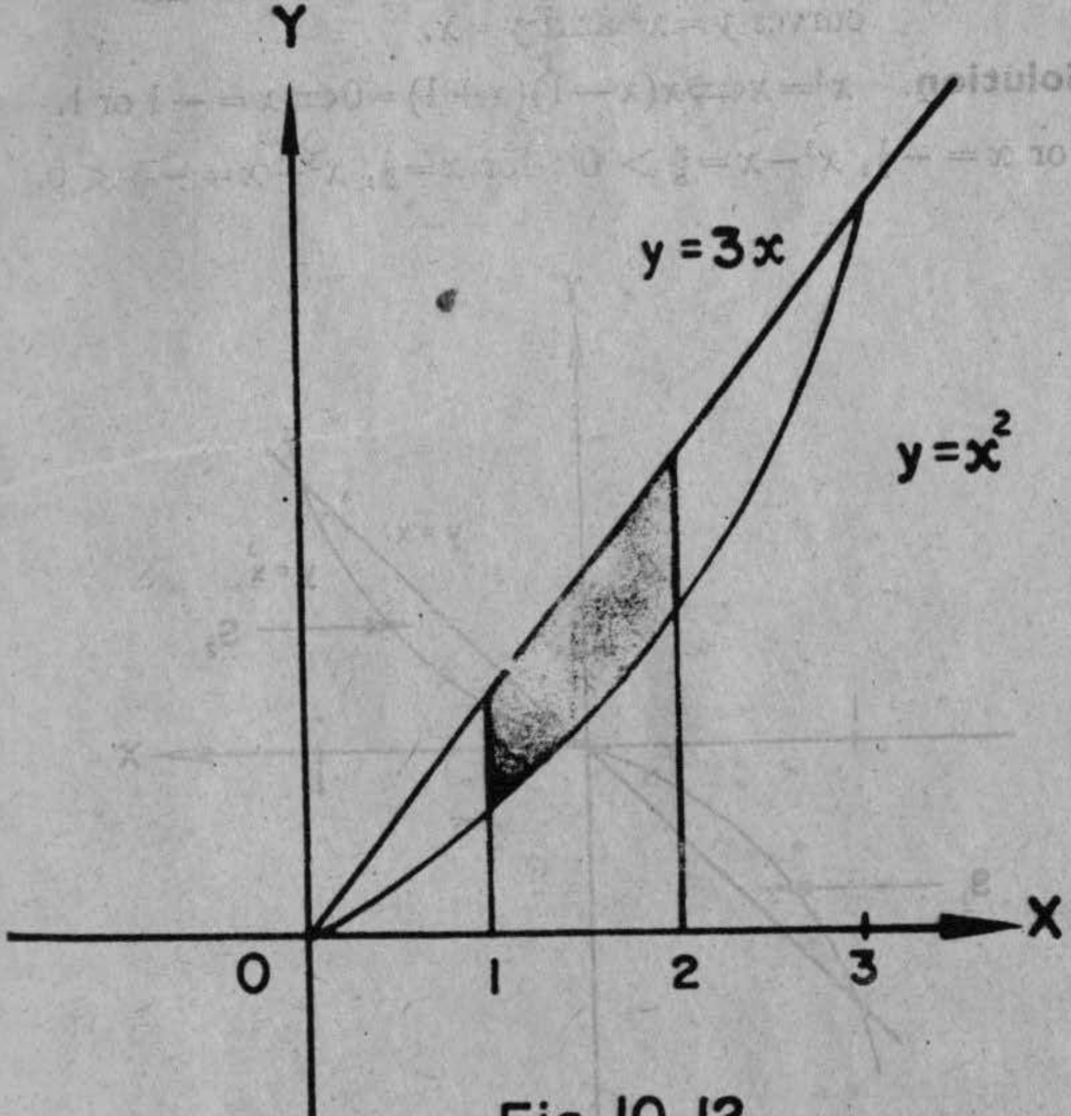
Solution. $3x = x^2 \Leftrightarrow x = 0$ or 3.

For $x = \frac{3}{2}$, $x^2 < 3x$.

Hence the area is described by

$$1 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 3x$$

and is the shaded part in Figure 10.12.



Fig·10.12

Hence the required area is given by

$$\int_1^2 (3x - x^2)\, dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_1^2$$

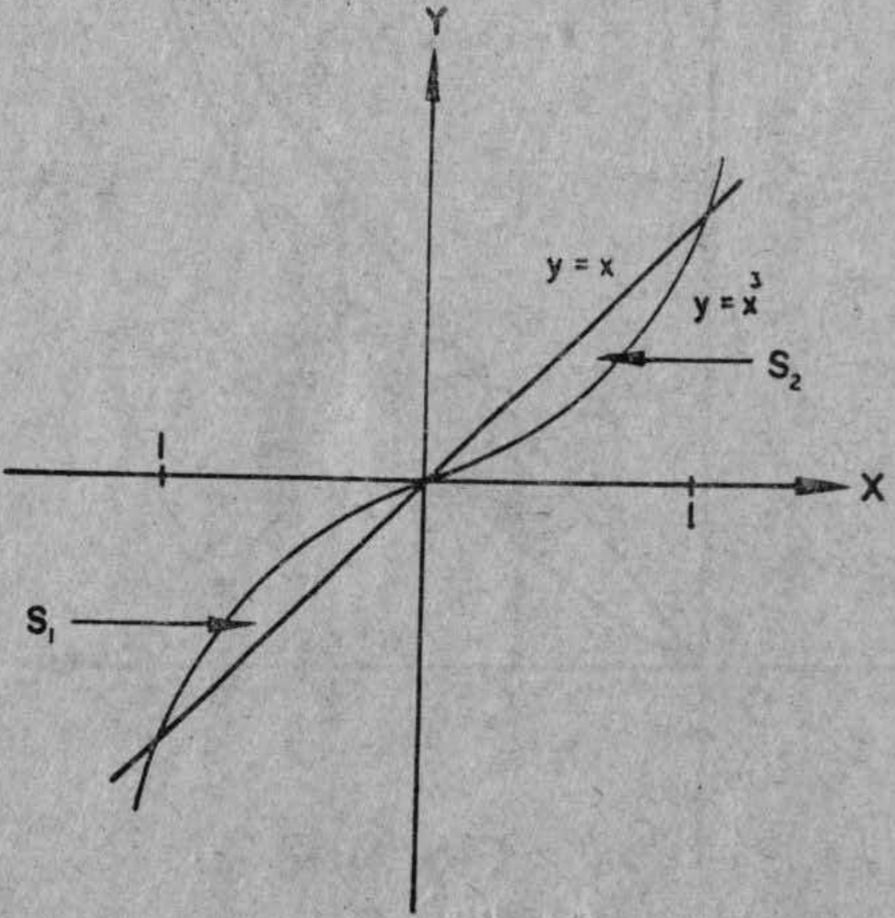
$$= (6 - \tfrac{8}{3}) - (\tfrac{3}{2} - \tfrac{1}{3}) = \tfrac{9}{2} - \tfrac{7}{3}$$

$$= \tfrac{13}{6}.$$

Example 3. Find the area enclosed between the curves $y = x^3$ and $y = x$.

Solution. $x^3 = x \Leftrightarrow x(x-1)(x+1) = 0 \Leftrightarrow x = -1$ or 1.

For $x = -\frac{1}{2}$, $x^3 - x = \frac{3}{8} > 0$; for $x = \frac{1}{2}$, $x^3 - x = -\frac{3}{8} < 0$.



Fig·10.13

Hence the required area is the area of the following two sets.

$$S_1 : -1 \leqslant x \leqslant 0,\ x \leqslant y \leqslant x^3$$
$$S_2 :\ 0 \leqslant x \leqslant 1,\ x^3 \leqslant y \leqslant x,$$

which are shown in figure 10.13.

Hence area of $S_1 = \int_{-1}^{0} (x^3 - x)\, dx$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0}$$
$$= 0 - [\tfrac{1}{4} - \tfrac{1}{2}] = \tfrac{1}{4}\ ;$$

and area of $S_2 = \int_{0}^{1} (x - x^3)\, dx$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1}{2} - \frac{1}{4}\right) - 0$$
$$= \frac{1}{4}.$$

Hence the area included between the curves

$$= S_1 + S_2$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Exercise 10.6

1. In the following equations find $s(t)$, given that the velocity of a particle along a line is $v(t)$ and $S(t_0) = S_o$:—

(*i*) $v(t) = 2t + 3,\ t_o = 1;\ s_o = 3$
(*ii*) $v(t) = \frac{3}{2}t^2 - 3t + 1,\ t_o = 2,\ s_o = 4$
(*iii*) $v(t) = t^2 + t - 2,\ t = -1,\ s_o = 2$
(*iv*) $v(t) = -t^2 + 2t + 1,\ t_o = 2,\ s_o = 0.$

2. A body moving in a straight line with constant acceleration a has initial velocity $v_0 = u$, *i.e.* $v = u$ at $t = 0$. Find v as a function of time 't'.

3. The acceleration for rectilinear motion is given by $a = 8 - 6t$, where $v = 3$, $s = 5$ at $t = 0$ Find v and s at $t = 1, 2, 3, 4, 5$.

4. In the following questions, find the area under the curve $y = f(x)$ from $x = a$ to $x = b$:—

(*i*) $f(x) = x^2 - 2x + 2$, $a = 1$, $b = 3$
(*ii*) $f(x) = 3x - x^2$, $a = 1$, $b = 2$
(*iii*) $f(x) = x^2 + x + 1$, $a = -2$, $b = 1$
(*iv*) $f(x) = 6 + x - x^2$, $a = -2$, $b = -1$.

5. Find the area in the first quadrant bounded by the parabola $y^2 = 4ax$ and the ordinate at $x = c$.

6. Find the area bounded by x-axis and one convolution of $y = \text{Sin } x$.

7. Find the area bounded by $a^2y = x^3$, the x-axis and the line $x = 2a$.

8. Find the area bounded by $y^2 - 3x + 3 = 0$ and the line $x = 4$.

9. Find the area enclosed by $x^2 + y^2 = a^2$.

10. Find the area enclosed by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

11. Find the area enclosed by the curves $y = x^2$ $y = 8 - x^2$ between the ordinates $x = -1$ and $x = 1$

12 Find the area enclosed by the curves $y = \sqrt{x+1}$, $y = \frac{1}{2}x$ between the ordinates $x = 0$ and $x = 3$.

13. Find the area enclosed between the curves :—

(*i*) $y = x^2$, $y^2 = x$
(*ii*) $y^2 = 4x + 1$, $x + y = 1$.

ANSWERS

1. (*i*) $s\ (t) = t^2 + 3t - 1$
   (*ii*) $s\ (t) = (\frac{1}{2})\ t^3 - (\frac{3}{2})\ t^2 + t + 4$
   (*iii*) $s\ (t) = (\frac{1}{3})\ t^3 + (\frac{1}{2})\ t^2 - 2t - \frac{1}{6}$
   (*iv*) $s\ (t) = -(\frac{1}{3})\ t^3 + t^2 + t - \frac{10}{3}$
2. $v = u + at,\ s = ut + \frac{1}{2}at^2$
3. $v = 8t - 3t^2 + 3,\ s = 4t^2 - t^3 + 3t + 5$
4. (*i*) $\frac{14}{3}$
   (*ii*) $\frac{13}{6}$
   (*iii*) $\frac{9}{2}$
   (*iv*) $\frac{13}{8}$
5. $\frac{4}{3}\sqrt{a}\ C^{\frac{3}{2}}$
6. 2
7. $4a^2$
8. 12
9. $\pi\ a^2$
10. $\pi\ a\ b$
11. $14\frac{2}{3}$
12. $\frac{29}{12}$
13 (*i*) $\frac{1}{3}$
   (*ii*) $\frac{98}{3}$

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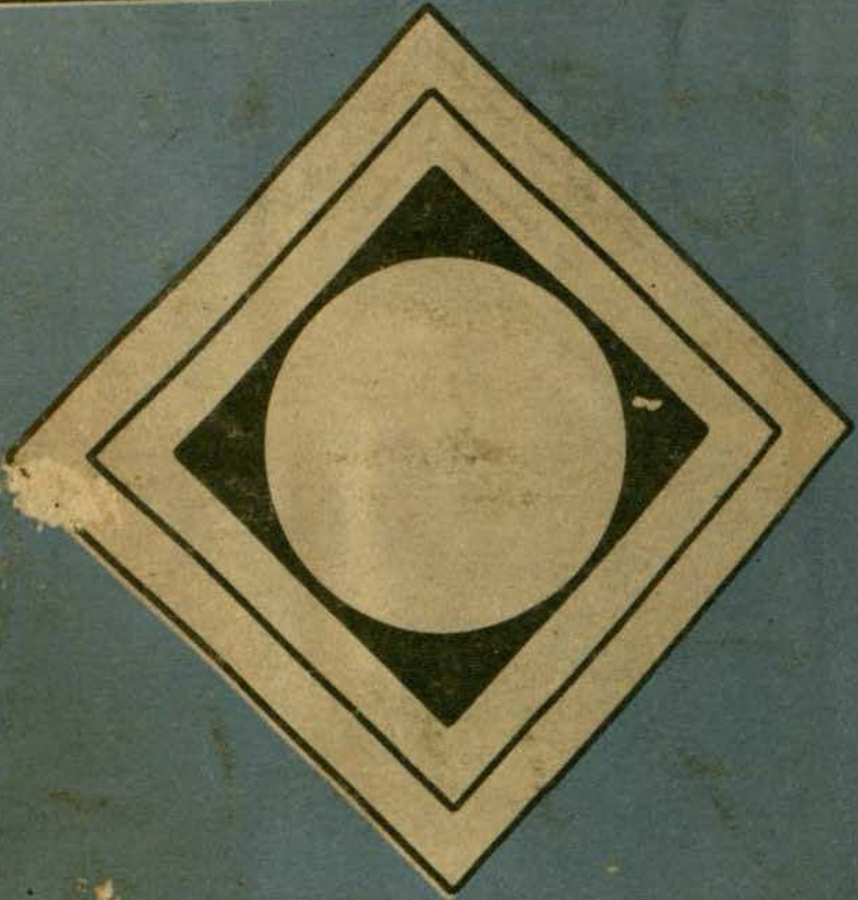

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